Math 14
Winter 2009
Monday, January 5

Functions from $\mathbb{R}^m$ to $\mathbb{R}^n$

We will be dealing with functions

$$F : \mathbb{R}^m \to \mathbb{R}^n.$$

Examples:

More examples:

A particularly important kind of function from $\mathbb{R}^m$ to $\mathbb{R}^n$ is given by matrix multiplication. Let A be an $n \times m$ matrix. This means A has n rows and m columns.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{1m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix}.$$

If $\vec{v}$ is an m-element column vector (that is, an $m \times 1$ matrix), the product $A\vec{v}$ is an n-element column vector (that is, an $n \times 1$ matrix). The entries of $A\vec{v}$ are the dot products of the rows of A with the vector $\vec{v}$.

$$A\vec{v} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{1m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m \end{pmatrix}.$$

A function $F : \mathbb{R}^m \to \mathbb{R}^n$ given by multiplication by a matrix with constant entries,

$$F(\vec{v}) = A\vec{v},$$

is called a *linear* function from $\mathbb{R}^m$ to $\mathbb{R}^n$.

Warning: This is a different meaning of "linear" than you are probably used to. In high school algebra, functions whose graphs are lines, functions of the form $f(x) = ax + b$, are generally called linear.

For us, a function $f(x) = ax$ is a linear function from $\mathbb{R}^1$ to $\mathbb{R}^1$. (We can think of the number a as a 1×1 matrix.) However, $f(x) = ax + b$ is *not* linear; it is the sum of a linear function ($g(x) = ax$) and a constant function ($h(x) = b$).

The sum of a linear function and a constant function is called an *affine* function. Tangent line approximations to functions $f : \mathbb{R}^1 \to \mathbb{R}^1$

$$T(x) = f'(x_0)(x - x_0) + f(x_0) = f'(x_0)(x) + (f'(x_0)(-x_0) + f(x_0)),$$

and their generalizations to functions from $\mathbb{R}^m$ to $\mathbb{R}^n$, are important affine functions.

We can also multiply an $n \times m$ matrix A by an $m \times p$ matrix B to get an $n \times p$ matrix AB. If the columns of B (written as column vectors) are $\vec{c}_1$, $\vec{c}_2$, ..., $\vec{c}_p$, then the columns of AB are $A\vec{c}_1$, $A\vec{c}_2$, ..., $A\vec{c}_p$. In other words, the entry in row i, column j of AB is the dot product of row i of A with column j of B.

Here are a few important facts about matrix multiplication. Matrix multiplication

— is associative.

— distributes over matrix addition, both on the left and on the right.

— has identity matrices.

— is not commutative.

— does not satisfy cancellation laws.

Some but not all $n \times n$ matrices have multiplicative inverses.

The determinant of an $n \times n$ matrix A, denoted $det(A)$ or $|A|$, is a number with a geometric significance: The absolute value of the determinant, $|det(A)|$ is the measure (area if $n = 2$, volume if $n = 3$, and so forth) of the n-dimensional parallelopiped (parallelogram if $n = 2$) with edges given by the rows (or columns) of A. If the determinant is positive, then the rows (or columns) of A, in order, are oriented positively (according to the right-hand rule for $n = 3$); if the determinant is negative, then have the opposite orientation.

Another use of the determinant is in computing the cross product of two vectors in $\mathbb{R}^3$. The vector

$$\vec{a} \times \vec{b} = (a_1, a_2, a_3) \times (b_1, b_2, b_3) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

is normal (perpendicular) to both $\vec{a}$ and $\vec{b}$, and $\vec{a} \times \vec{b}$, $\vec{a}$, and $\vec{b}$ in that order are positively oriented. Furthermore, $|\vec{a} \times \vec{b}|$ is the area of the parallelogram with edges given by $\vec{a}$ and $\vec{b}$.

The same thing is true in general: If $\vec{a}_1$, $\vec{a}_2$, $\ldots$, $\vec{a}_{n-1}$ are $n-1$ many vectors in $\mathbb{R}^n$, then the vector given by the determinant of the matrix whose first row consists of the standard basis vectors $\hat{e}_1$, $\hat{e}_2$, $\ldots$, $\hat{e}_n$ of $\mathbb{R}^n$ and whose remaining rows are $\vec{a}_1$, $\vec{a}_2$, $\ldots$, $\vec{a}_{n-1}$,

$$\begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \cdots & \hat{e}_{n-1} & \hat{e}_n \\ & & \vec{a}_1 & & \\ & & \vdots & & \\ & & \vec{a}_{n-1} & & \end{vmatrix}$$

is normal to $\vec{a}_1$, $\ldots$, $\vec{a}_{n-1}$, gives a positively oriented collection if added to the beginning of the list, and has length equal to the measure of the $n-1$-dimensional parallelopiped with edges given by $\vec{a}_1$, $\ldots$, $\vec{a}_{n-1}$.

For example, if $n = 2$, and $\vec{a} = (a_1, a_2)$, then

$$\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} \\ a_1 * a_2 & \end{vmatrix} = (a_2, -a_1)$$

is normal to $\vec{a}$, $\vec{b}$ and $\vec{a}$ in that order form a positively oriented system (the direction from $\vec{b}$ to $\vec{a}$ is counterclockwise), and the length of $\vec{b}$ is equal to the length of $\vec{a}$.