Math 14 Winter 2004

Calculus of Vector-valued Functions, Honors

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Homework Project I (20 points)

Vector Calculus and Integral Geometry

Due date: Monday, February 9, at the end of the lecture time

Project objectives

Give a brief glance into deep relation between vector calculus and geometry. Guide through a sequence of small steps to discovery of one of the fundamental identities of integral geometry.

Description of the project

Let $Q = [0, \pi] \times \mathbb{R} = \{(\varphi, r) \in \mathbb{R}^2 \,|\, 0 \le \varphi \le \pi, r \in \mathbb{R}\}$ be an (infinite) rectangle in the plane. For any point $(\varphi, r) \in Q$ from this rectangle, denote by $L(\varphi, r)$ the line defined in the following way: mark the point $(r, 0)$ on the x-axis, rotate the axis by φ in the counter-clockwise direction, and draw a line through the marked point perpendicular to the rotated axis (see Figure 1). It is obvious that $L(0, r)$ and $L(\pi, -r)$ are the same lines.

1 Check that the map $(\varphi, r) \mapsto L(\varphi, r)$ is onto, that is, one obtains every line in the plane as $L(\varphi, r)$ for some (φ, r). Given a line p, explain how to (geometrically) find φ_p and r_p, so that $p = L(\varphi_p, r_p)$.

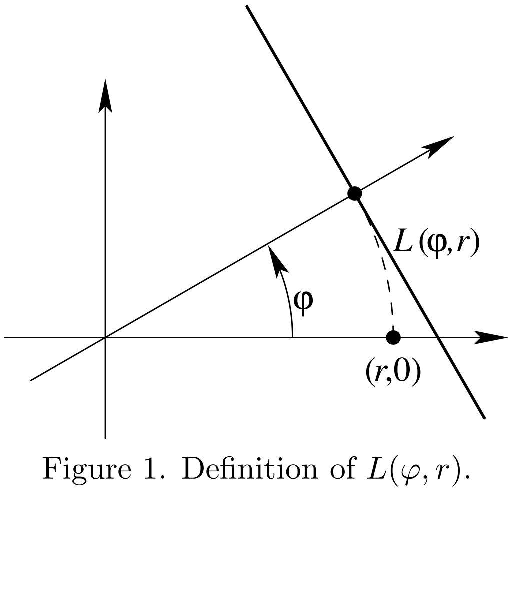

Figure 1. Definition of $L(\varphi, r)$.

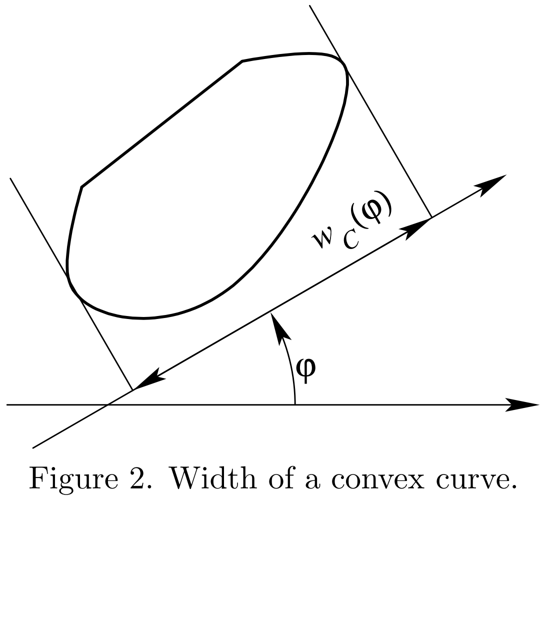

Figure 2. Width of a convex curve.

Consider now a curve C in the plane parametrized by $\mathbf{c} : [a, b] \to \mathbb{R}^2$. We assume that $\mathbf{c}$ is **piecewise smooth**, that is, $\mathbf{c}$ is continuous and there exist $a = t_0 < t_1 < \cdots < t_n = b$, so that $\mathbf{c}$ is smooth on every $[t_i, t_{i+1}]$. To be **smooth on an interval** means that the derivative $\mathbf{c}'(t)$ exists, continuous, and is not zero everywhere in this interval (except, possibly, the endpoints).

Denote by $n(\varphi, r)$ the number $\#(C \cap L(\varphi, r))$ of intersection points of C and the line $L(\varphi, r)$. Note that this number can be infinite (how?). Define the function $f_C : Q \to \mathbb{Z} \subset \mathbb{R}$ as follows:

$$f_C(\varphi, r) = \begin{cases} n(\varphi, r) & \text{if } n(\varphi, r) < \infty; \\ 0 & \text{otherwise.} \end{cases}$$

(2) Prove that f_C is zero outside of some (finite) rectangle. More precisely, find some α and β such that $f_C(\varphi, r) = 0$ if $r < \alpha$ or $r > \beta$. Then the rectangle in question is $[0, \pi] \times [\alpha, \beta]$. Denote it by R.

(3) Check that f_C is continuous almost everywhere and find all points where it is not. In more details: assume first that $\mathbf{c}$ is smooth on the whole $[a, b]$. Find geometrical conditions on $L(\varphi, r)$ such that f_C is not continuous at (φ, r). Describe explicitly (by a formula in terms of $\mathbf{c}$) all points (φ, r) such that f_C is not continuous at these points. What changes if $\mathbf{c}$ is piecewise smooth?

(4) Conclude (using Theorem 2 from section 5.2) that f_C is integrable over R. Define the double integral of f_C over Q (denoted by $I(C)$) as

$$I(C) = \iint\limits_Q f_C(\varphi, r) dr\, d\varphi = \iint\limits_R f_C(\varphi, r) dr\, d\varphi$$

(5) Prove that $I(C)$ is additive with respect to taking union. That is, if C_1 and C_2 are two curves and $C = C_1 \cup C_2$ then $I(C) = I(C_1) + I(C_2)$.

(6) Let C be a straight line segment between two points. Prove that $I(C) = 2\,\text{length}(C)$. You can do this in steps: first prove this formula for a segment between $(0, 0)$ and $(x, 0)$ and then check that the integral doesn't change if you rotate and/or translate this segment.

(7) Let C be a **polyline**, that is, a chain of line segments joined at the endpoints. Prove that $I(C) = 2\,\text{length}(C)$.

(8) Let C be an arbitrary piecewise smooth curve. Conclude that

$$\iint\limits_Q f_C(\varphi, r) dr\, d\varphi = 2\,\text{length}(C)$$

9 Let now C be a **closed convex curve**, that is, a closed curve such that the segment connecting any two points on it is completely contained in the region bounded by the curve. An example of a convex curve is shown in Figure 2. Denote by $w_C(\varphi)$ the width of C in the direction of φ (see Figure 2). Deduce from the previous formula that

$$\int_0^\pi w_C(\varphi)d\varphi = \text{length}(C)$$

Closing remarks

1. The equation from step 8 has the name of **Crofton Theorem** and can be formulated as follows: *the length of a plane curve C is equal to half of the integral over the set of all lines in the plane of number of intersection points between these lines and the curve C.*
2. Although the parametrization $\mathbf{c}$ of C was used to prove integrability of f_C, it doesn't appear in the final formula from step 8 for the length of C. Therefore, we can work with "geometrical" curves only, that is, with curves presented as subsets of the plane. They can be closed or not, have corners, and even consist of several disjoint or intersecting pieces, the formula will still hold true. We only need to know that the curve in question is **parametrizable**. The actual parametrization doesn't matter really.
3. The formula from step 9 is very convenient for approximating the length of a closed convex curve using a ruler only. What you need to do is to measure the width of the curve in several different directions and form the Riemann sum. Try to use this method to approximate the length of different curves.
4. The formulas discovered in this project are first is a series of fascinating results that belong to **Integral Geometry**, classical but still fast developing area of mathematics. For example, if one integrates not the number of intersection points, but the total length of the intervals that the curve cuts from the lines, then the result is proportional to the area of the region bounded by the curve. Formulas like that have numerous applications in image recognition, tomography and other areas.
5. For more information on this subject please refer to classical books by Luis Santalo "Introduction to integral geometry" (relatively easy) and "Integral geometry and geometric probability" (more advanced). For some related problems see also
   `http://mathworld.wolfram.com/BertrandsProblem.html`

Regulations concerning this project

1. Students can work on this project individually or in groups of **up to 3 persons.**
2. In case a group of students decides to work on the project together, **discussion of the problem and common solving inside of such a group is encouraged**. Members of different groups **should not share any essential information about the solution.**
3. In case a group of students decides to work together on the project, they are allowed to submit one solution for the group. This would imply that all the students in the group **do understand the presented solution** and can explain it to the professor in charge if asked to do so. In case you decide to submit one solution for all the members of the group please write clearly the names of all the group members on top of the solution. If one solution is submitted on behalf of a group of (up to 3) students, the grade for the project shall be **the same for all the members of the group.**
4. Students are allowed and encouraged **to consult mathematical books** and the Professors in charge.
5. The project is due by **Monday, February 9**, before the end of the lecture time. By that time the solution should be presented in **typed or clearly written form** to the Professor in charge. The solution should be presented **with full details**. Presenting only the answer is not sufficient.