Math 13 Winter 2015 Your name: ____________________

Instructor (please circle): □ Pierre Clare □ Mingzhong Cai □ Erik van Erp

Problem set 8, due Wed Mar 4

Please show your work. No credit is given for solutions without justification.

(1) Evaluate the surface integral $\iint_S y^2\,dS$, if S is the part of the sphere $x^2+y^2+z^2=4$ with $z \geq \sqrt{3}$.

(2) Evaluate $\iint_S \mathbf{F}\cdot d\mathbf{S}$, if $\mathbf{F}=\langle x,y,5\rangle$ and S is the boundary of the solid region bounded by the cylinder $x^2+z^2=1$ and the planes $y=0$ and $x+y=2$.

(3) Evaluate $\iint_S \mathrm{curl}\mathbf{F}\cdot d\mathbf{S}$, if $\mathbf{F}=\langle e^{xy}, e^{xz}, x^2z\rangle$ and S is the half of the ellipsoid $4x^2+y^2+4z^2=4$ that lies to the right of the xz-plane, oriented in the direction of the positive y-axis.

① $\vec{r}(\varphi,\theta) = (2\sin\varphi\cos\theta,\ 2\sin\varphi\sin\theta,\ 2\cos\varphi)$

$0 \le \varphi \le \frac{\pi}{6}, \quad 0 \le \theta \le 2\pi.$

from textbook, $|\vec{r_\varphi}\times\vec{r_\theta}| = 4\sin\varphi$.

$$\text{so } \iint_S y^2 dS = \int_0^{2\pi}\int_0^{\pi/6} (2\sin\varphi\sin\theta)^2 \cdot 4\sin\varphi\, d\varphi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/6} 16\sin^3\varphi\sin^2\theta\, d\varphi\, d\theta$$

$$= 16\cdot\int_0^{2\pi}\sin^2\theta\, d\theta\left(-\int_0^{\pi/6}(1-\cos^2\varphi)\, d(\cos\varphi)\right)$$

$$= -16\int_0^{2\pi}\frac{1-\cos 2\theta}{2}\, d\theta \cdot \left(\cos\varphi - \frac{1}{3}\cos^3\varphi\right)\Big|_{\varphi=0}^{\varphi=\pi/6}$$

$$= -16\pi\cdot\left[\left(\frac{\sqrt{3}}{2}-\frac{1}{3}\cdot\frac{3}{4}\cdot\frac{\sqrt{3}}{2}\right)-\left(1-\frac{1}{3}\right)\right]$$

$$= -16\pi\cdot\left(\frac{3\sqrt{3}}{8}-\frac{2}{3}\right)$$

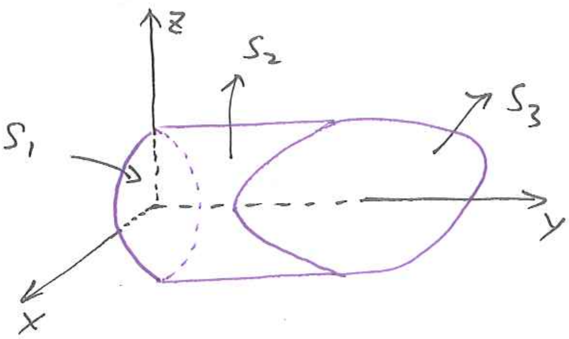
$$= \left(\frac{32}{3}-6\sqrt{3}\right)\pi$$

② The surface S has three parts:

S_1 : left face (bottom face, $y=0$).

S_2 : side face.

S_3 : right face ($x+y=2$)



on S_1, use x and z as parameters.

$y=0$, $F=(x,y,5)$. normal vector $\vec{n} = -\vec{j}$.

$$\iint_{S_1} \vec{F}\cdot d\vec{S} = -\iint_D \left(-P\frac{\partial y}{\partial x} + Q - R\frac{\partial y}{\partial z}\right) dA = -\iint_D 0\, dA = 0.$$

on S_3, similarly $y = 2-x$, $F=(x,y,5)$. $\vec{n} = \vec{i} + \vec{j}$

$$\iint_{S_3} \vec{F}\cdot d\vec{S} = \iint_D \left(-P\frac{\partial y}{\partial x} + Q - R\frac{\partial y}{\partial z}\right) dA = \iint_D (-x(-1) + (2-x)\, dA$$

$$= \iint_D 2\, dA = 2A(D) = 2\pi.$$

on S_2. $\vec{r}(u,v) = (\cos u, v, \sin u)$ where $0 \le u \le 2\pi$, $0 \le v \le 2-\cos u$.

outward normal vector $\vec{n} = (\cos u, 0, \sin u)$. so

$$\iint_{S_2} \vec{F}\cdot d\vec{S} = \iint_{S_2} \vec{F}\cdot\vec{n}\, dS = \int_0^{2\pi}\int_0^{2-\cos u} (\cos u, v, 5)\cdot(\cos u, 0, \sin u)\, dv\, du$$

$$= \int_0^{2\pi}\int_0^{2-\cos u} \cos^2 u + 5\sin u\, dv\, du$$

$$= \int_0^{2\pi} \cos^2 u(2-\cos u) + 5\sin u(2-\cos u)\, du$$

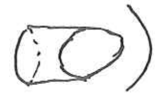
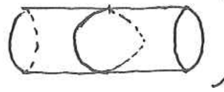
$$= \int_0^{2\pi} (2\cos^2 u - \cos^3 u + 10\sin u - 5\sin u\cos u)\, du$$

$$= 2\pi$$

so $\iint_S \vec{F}\cdot d\vec{S} = 2\pi + 0 + 2\pi = 4\pi$.

Remark: using divergence theorem,

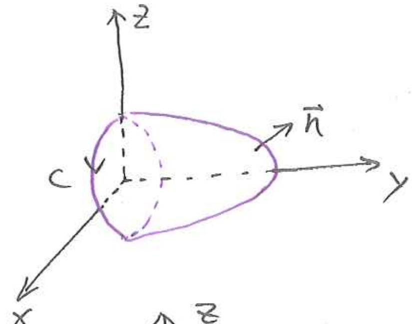
$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_E \operatorname{div}\vec{F}\, dV = \iiint_E 2\, dV = 2\cdot V(E)$$

$= 2\cdot V($  $) = V($  $) = \pi\cdot 1^2\cdot 4 = 4\pi.$

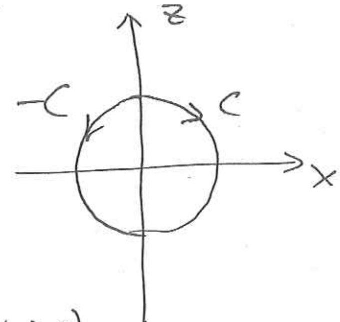
③ by Stokes' theorem,

$$\iint_S \operatorname{curl} \vec{F} \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r}$$

where C is shown on the graph.



$-C$ is the curve $\vec{r}(\theta) = (\cos\theta, 0, \sin\theta)$ $0 \le \theta \le 2\pi$.



so $$\iint_S \operatorname{curl} \vec{F} \cdot d\vec{S} = -\int_{-C} \vec{F} \cdot d\vec{r}$$

$$= -\int_{-C} P\,dx + Q\,dy + R\,dz$$

$$= -\int_0^{2\pi} e^0\, d(\cos\theta) + 0 + \cos^2\theta \sin\theta\, d(\sin\theta)$$

$$= -\int_0^{2\pi} (-1 + \cos^3\theta) \sin\theta\, d\theta$$

$$= -\cos\theta + \frac{1}{4}\cos^4\theta \Big|_{\theta=0}^{\theta=2\pi}$$

$$= 0.$$