Math 13 Winter 2015 Your name: ____________________

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Problem set 6, due Wed Feb 18

Please show your work. No credit is given for solutions without justification.

(1) For each of the two vector fields below, calculate the curl and the divergence. If the vector field is conservative, find a potential function.

$$\mathbf{F}(x,y,z) = \langle xyz^2, x^2yz^2, x^2y^2z\rangle \quad \mathbf{G}(x,y,z) = \langle 1, \sin z, y\cos z\rangle$$

(2) Let C be the segment of the helix $\mathbf{r}(t) = (\cos t, \sin t, t)$ with $0 \le t \le 2\pi$. Evaluate the line integral $\int_C \mathbf{G}\cdot d\mathbf{r}$, where $\mathbf{G}$ is the vector field from problem 1.

(3) If C is the triangle from $(0,0)$ to $(1,1)$ to $(0,1)$ to $(0,0)$, evaluate the line integral

$$\int_C \sqrt{x^2+1}\,dx + \arctan(x)\,dy$$

① $\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz^2 & x^2yz^2 & x^2y^2z \end{vmatrix} = \vec{i}\left(\frac{\partial(x^2y^2z)}{\partial y} - \frac{\partial(x^2yz^2)}{\partial z}\right)$

$+ \vec{j}\left(\frac{\partial(xyz^2)}{\partial z} - \frac{\partial(x^2y^2z)}{\partial x}\right) + \vec{k}\left(\frac{\partial(x^2yz^2)}{\partial x} - \frac{\partial(xyz^2)}{\partial y}\right)$

$= (2x^2yz - 2x^2yz)\vec{i} + (2xyz - 2xy^2z)\vec{j} + (2xyz^2 - xz^2)\vec{k}$

so $\vec{F}$ is not conservative.

$\operatorname{div}\vec{F} = yz^2 + x^2z^2 + x^2y^2.$

$\operatorname{curl}\vec{G} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 1 & \sin z & y\cos z \end{vmatrix} = \vec{i}(\cos z - \cos z) + \vec{j}(0-0) + \vec{k}(0-0) = \vec{0}.$

so $\vec{G}$ is conservative. To find a potential function, we solve:

$$\begin{cases} \frac{\partial g}{\partial x} = 1 & ① \\ \frac{\partial g}{\partial y} = \sin z & ② \\ \frac{\partial g}{\partial z} = y\cos z & ③ \end{cases}$$

① $\Rightarrow$ $g(x,y,z) = x + h(y,z)$. and by ② + ③ we have

$$\begin{cases} \frac{\partial h}{\partial y} = \sin z \\ \frac{\partial h}{\partial z} = y\cos z. \end{cases} \quad \text{and } h(y,z) = y\sin z + C.$$

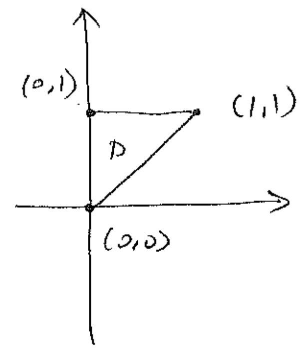
so $g(x,y,z) = x + y\sin z + C$

$$\operatorname{div} \vec{G} = 0 + 0 + y(-\sin z) = -y\sin z.$$

②. $\vec{G}$ is conservative, so by FT.

$$\int_C \vec{G}\cdot d\vec{r} = g(\vec{r}(2\pi)) - g(\vec{r}(0)) = g(1,0,2\pi) - g(1,0,0) = 0.$$

③. by Green's Theorem, $\int_C \sqrt{x^2+1}\, dx + \arctan x\, dy$



$$= \iint_D \left(\frac{\partial(\arctan x)}{\partial x} - \frac{\partial(\sqrt{x^2+1})}{\partial y} \right) dA$$

$$= \iint_D \frac{1}{x^2+1}\, dA$$

$$= \int_0^1 \int_x^1 \frac{1}{x^2+1}\, dy\, dx$$

$$= \int_0^1 \frac{1}{x^2+1} - \frac{x}{x^2+1}\, dx$$

$$= \arctan(x)\Big|_0^1 - \frac{1}{2}\ln|x^2+1|\Big|_0^1$$

$$= \arctan 1 - \frac{1}{2}\ln 2$$

$$= \frac{\pi}{4} - \frac{1}{2}\ln 2.$$