Math 13 Winter 2015 Your name: ____________________

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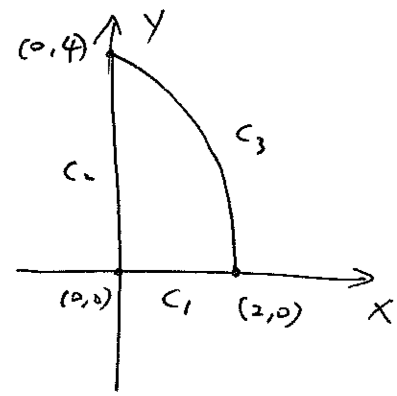
Problem set 5, due Wed Feb 11

Please show your work. No credit is given for solutions without justification.

(1) Evaluate the line integral $\int_C x\,ds$ if C is the closed curve in $\mathbb{R}^2$ consisting of the straight line segment from $(0,0)$ to $(2,0)$, the straight line segment from $(0,0)$ to $(0,4)$, and the segment of the parabola $y = 4 - x^2$ that connects $(0,4)$ to $(2,0)$.

(2) Evaluate the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ for the vector field $\mathbf{F}(x,y,z) = \langle x+z, y-x, x^2\rangle$ and the straight line segment C from $(1,0,0)$ to $(0,1,1)$.

(3) Let $\mathbf{F} = \nabla f$ be the gradient vector field of the function $f(x,y) = \tan(x^2+y^2)$. If C is the arc of the circle $\mathbf{r}(t) = (5\cos(t), 5\sin(t))$ with $\frac{\pi}{4} \le t \le \frac{3\pi}{4}$, is the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ positive, negative, or zero? Explain your answer.



1 $\quad C_1: \quad x(t) = t, \quad y(t) = 0, \qquad 0 \le t \le 2.$

$$\int_{C_1} x\,ds = \int_0^2 t\cdot\sqrt{1^2+0^2}\cdot dt = \int_0^2 t\,dt = 2$$

$C_2: \quad x(t) = 0, \quad y(t) = t. \qquad 0 \le t \le 4.$

$$\int_{C_2} x\,ds = \int_0^4 0\cdot ds = 0$$

$C_3: \quad x(t) = t, \quad y(t) = 4 - t^2. \qquad 0 \le t \le 2.$

$$\int_{C_3} x\,ds = \int_0^2 t\sqrt{1^2 + (-2t)^2}\,dt = \int_0^2 t\sqrt{1+4t^2}\,dt$$

$$= \frac{1}{8}\int_0^2 \sqrt{1+4t^2}\,d(1+4t^2)$$

$$= \frac{1}{8}\cdot\frac{2}{3}\cdot(1+4t^2)^{\frac{3}{2}}\Big|_{t=0}^{t=2}$$

$$= \frac{1}{12}\left(17\sqrt{17} - 1\right)$$

so $\displaystyle\int_C x\,ds = 2 + \frac{1}{12}\left(17\sqrt{17}-1\right)$

(2) $C: \quad x(t) = 1-t, \quad y(t) = t, \quad z(t) = t, \qquad 0 \le t \le 1.$

$$\int_C \vec{F} \cdot d\vec{r} = \int_C (x+z)\,dx + (y-x)\,dy + x^2\,dz$$

$$= \int_0^1 1 \cdot (-dt) + (2t-1)\,dt + (t^2-2t+1)\,dt$$

$$= \int_0^1 (t^2-1)\,dt$$

$$= \frac{1}{3} - 1$$

$$= -\frac{2}{3}$$

(3) $f(x,y) = \tan(x^2+y^2)$

$$\vec{\nabla} f = \sec^2(x^2+y^2) \cdot 2x\,\vec{i} + \sec^2(x^2+y^2) \cdot 2y\,\vec{j}.$$

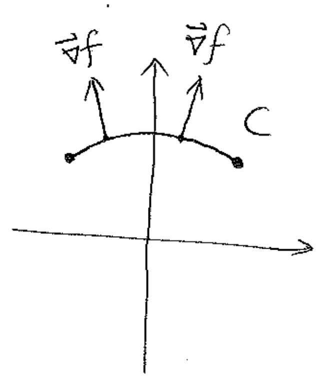
$$= 2\sec^2(x^2+y^2)\left(x\vec{i} + y\vec{j}\right)$$

so $\vec{\nabla} f$ at any point (x,y) is either pointing towards the origin, or away from the origin.

That is. $\vec{\nabla} f(\vec{x})$ has the same (or opposite) direction as $\vec{x}$.

Now along the curve C, it is easy to see that $\vec{\nabla} f$ is always perpendicular to the curve C.

so the work $\int_C \vec{F} \cdot d\vec{r}$ is 0.



Alternative solution:

use fundamental theorem of line integrals

$$\int_C \vec{F} \cdot d\vec{r} = f(\vec{r}(t))\Big|_{t=\pi/4}^{t=3\pi/4}$$

$$= \tan 25 - \tan 25 \quad (x^2+y^2=25 \text{ on the circle})$$

$$= 0$$