MATH 13, WINTER 2011
WRITTEN HOMEWORK #7—SOLUTIONS

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This assignment will be due on Wednesday, February 23 at 12:30 p.m. in the boxes outside 105 Kemeny. Look for the column of boxes labeled "Math 13, Winter 2011" and put your assignment in the left ("IN") column corresponding to the first letter of your family name (A-F, G-M, N-S, T-Z).

Remember to show your work. A correct answer with no work shown will receive minimal credit. Your solutions should be detailed enough that any of your classmates could understand them simply by reading them.

(1) (17.4, #22) Let D be a region bounded by a simple closed path C in the xy-plane. Use Green's Theorem to prove that the coordinates of the centroid $(\bar{x}, \bar{y})$ of D are $\bar{x} = \frac{1}{2A} \oint_C x^2\, dy$ and $\bar{y} = -\frac{1}{2A} \oint_C y^2\, dx$ where A is the area of D.

(2) Let $\mathbf{F} = xy^2\mathbf{i} + (y+x)\mathbf{j}$. Integrate $(\nabla \times \mathbf{F}) \cdot \mathbf{k}$ over the region in the first quadrant bounded by the curves $y = x^2$ and $y = x$.

(3) (17.5, #25 plus some) Prove the following statements assuming that the appropriate partial derivatives exist and are continuous.
  (a) $\text{div}\,(f\mathbf{F}) = f\text{div}\,\mathbf{F} + \mathbf{F} \cdot \nabla f$
  (b) If $\mathbf{c}$ is a constant vector and $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, then curl $\frac{1}{2}(\mathbf{c} \times \mathbf{r}) = \mathbf{c}$.

(4) (17.5, #33) Use Green's Theorem in the form of Equation 13 to prove Green's first identity:

$$\iint_D f\nabla^2 g\, dA = \oint_C f(\nabla g) \cdot \mathbf{n}\, ds - \iint_D \nabla f \cdot \nabla g\, dA$$

where D and C satisfy the hypotheses of Green's Theorem and the appropriate partial derivatives of f and g exist and are continuous.

(5) (17.5, #20) Is there a vector field $\mathbf{G}$ on $\mathbb{R}^3$ such that curl $\mathbf{G} = \langle xyz, -y^2z, yz^2\rangle$? Explain your answer.

(6) (17.6, #26) Find a parametric representation for the part of the plane $z = x + 3$ that lies inside the cylinder $x^2 + y^2 = 1$.

(7) (17.7, #10) Evaluate the surface integral $\iint_S \sqrt{1 + x^2 + y^2}\,dS$, where S is the helicoid with the vector equation $\mathbf{r}(u, v) = u\cos v\mathbf{i} + u\sin v\mathbf{j} + v\mathbf{k}$, $0 \le u \le 1$, $0 \le v \le \pi$.

Suggested problems: 17.4: 6-9; 17.5: 7-8, 9-11, 12, 13-14, 22, 38; 17.6: 3-6, 13-18, 19-21; 17.7: 5-8