Math 13 Worksheet #8: Change of variables and the Jacobian

(1) Use the transformation $x = \frac{1}{4}(u + v)$, $y = \frac{1}{4}(v - 3u)$ to evaluate the integral

$$\iint_R (4x + 8y)\,dA,$$

where R is the parallelogram with vertices $(-1, 3)$, $(1, -3)$, $(3, -1)$, and $(1, 5)$.

1st. solve for u & v so we can transform vertices to get S.

$$① \quad 4x = u + v$$
$$② \quad 4y = -3u + v$$

① - ② yields $4x - 4y = 4u \Rightarrow u = x - y$

Use ① to solve for v. $V = 4x - u = 4x - x + y = 3x + y$

R	S	
$(-1,3)$	$(-1-3,\ -3+3) = (-4, 0)$	$-4 \le u \le 4$
$(1,-3)$	$(1+3,\ 3-3) = (4, 0)$	$0 \le v \le 8$
$(3,-1)$	$(3+1,\ 9-1) = (4, 8)$	
$(1,5)$	$(1-5,\ 3+5) = (-4, 8)$	

$$J = \begin{vmatrix} \frac{dx}{du} & \frac{dx}{dv} \\ \frac{dy}{du} & \frac{dy}{dv} \end{vmatrix} = \begin{vmatrix} 1/4 & 1/4 \\ -3/4 & 1/4 \end{vmatrix} = 1/16 + 3/16 = 1/4$$

$$\iint_{R} (4x+8y)\,dA = \frac{1}{4}\int_0^8 \int_{-4}^{4} (u + v + 2v - 6u)\,du\,dv = \frac{1}{4}\int_0^8 \int_{-4}^{4} (-5u + 3v)\,du\,dv$$

$$= \frac{1}{4}\int_0^8 -\frac{5u^2}{2} + 3uv \Big|_{-4}^{4} dv = \frac{24}{4}\int_0^8 v\,dv = \frac{6}{2}(8)^2 = 3 \cdot 64$$

(2) By making an appropriate change of variables, evaluate the integral $\iint\limits_R \sin(9x^2+4y^2)dA$, where R is the region in the first quadrant bounded by the ellipse $9x^2+4y^2=1$.

We want to make the ellipse transform into a circle.

$\Rightarrow 9x^2 = u^2 \Rightarrow u = 3x$ or $x = u/3$

$4y^2 = v^2 \Rightarrow v = 2y$ or $y = v/2$

So: $u^2+v^2=1$

$$J = \begin{vmatrix} 1/3 & 0 \\ 0 & 1/2 \end{vmatrix} = 1/6$$

$$\iint\limits_R \sin(9x^2+4y^2)\,dA = \iint\limits_S \sin(u^2+v^2)\left(\frac{1}{6}\right)dA$$

$$= \frac{1}{6}\int_0^{2\pi}\int_0^1 \sin(r^2)\, r\,dr\,d\theta$$

Switch to polar

$u = r^2$
$du = 2r\,dr$

$$= \frac{1}{6}\left(\frac{1}{2}\right)\int_0^{2\pi}\int_0^1 \sin u\,du\,d\theta$$

$$= \frac{1}{12}\int_0^{2\pi} -\cos u\Big|_0^1\,d\theta = \frac{\pi}{6}(1-\cos 1)$$