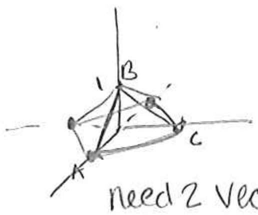
Math 13 Worksheet #5: Triple integrals and cylindrical coordinates

(1) Use a triple integral to find the volume of a pyramid whose base is the square with vertices $(1,0,0)$, $(0,1,0)$, $(-1,0,0)$, and $(0,-1,0)$ and whose top vertex is $(0,0,1)$.



Note: It is symmetric so we only need to compute the volume in  and multiply by 4

Now we need the plane making the face of the pyramid.

need 2 vectors $\overline{AB} = \langle -1,0,1\rangle$ $\overline{AC} = \langle -1,1,0\rangle$

$\bar{n} = \overline{AC} \times \overline{AB} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = \bar{i} + \bar{j} + \bar{k}$

Plane. $\bar{n} \cdot (\langle x,y,z\rangle - A) = 0 = (x-1) + y + z = 0 \Rightarrow z = 1 - x - y$

continue on page 2

(2) Find the center of mass of the pyramid assuming the density is uniform inside.

Uniform density $\Rightarrow$ $\rho(x,y,z) = k$ - constant.

Since density is uniform & the base is symmetric in both x & y direction $\bar{x} = \bar{y} = 0$.

Now we need to compute $\bar{z}$.

Mass $= M = k \iiint_B dV = \frac{k}{6}$ by 1

$M_{xy} = 4k \int_0^1 \int_0^{1-x} \int_0^{1-x-y} z \, dz \, dy \, dx$

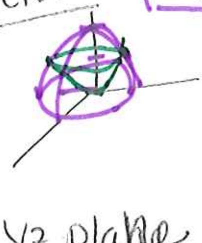
$= 4k \int_0^1 \int_0^{1-x} \frac{z^2}{2} \Big|_0^{1-x-y} dy \, dx$

continue on page 2

(3) Find the mass of the slice of the right circular cylinder $x^2 + z^2 = 4$ bounded on the left by the xz–plane and the on right by the plane with equation $x - y + z = -4$ if the density at each point in the cylinder is proportional to the distance of the point to the xz–plane. (You can choose cylindrical or Cartesian coordinates.) see page 3

(4) Find the volume of the solid that lies between the paraboloid $\boxed{z = x^2 + y^2}$ and the sphere $\boxed{x^2 + y^2 + z^2 = 4}$

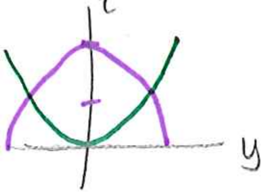
3D Picture



What are the surfaces in cylindrical coordinates?

$z = r^2$ $\quad r^2 + z^2 = 4 \Rightarrow z = \sqrt{4 - r^2}$

In yz plane



What is the intersection?

$r^2 + r^4 = 4 \Rightarrow r^4 + r^2 - 4 = 0$

$r^2 = \frac{-1 \pm \sqrt{16-4}}{2} = -\frac{1}{2} \pm \frac{\sqrt{12}}{2}$

$= -\frac{1}{2} \pm \sqrt{3}$

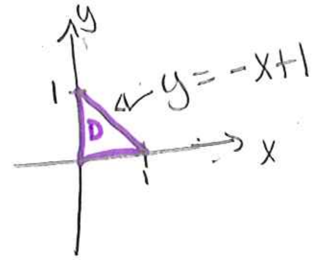
negative does not make sense

$\Rightarrow r^2 = -\frac{1}{2} + \sqrt{3} \Rightarrow r = (\sqrt{3} + 1/2)^{1/2}$ see page 4

(1) Continued

$$V = 4\iiint_B 1\,dV = 4\iint_D \int_0^{1-x-y} dz\,dA.$$

What is D? D lives in xy-plane.



$$V = 4\int_0^1 \int_0^{1-x} \int_0^{1-x-y} dz\,dy\,dx$$

$$= 4\int_0^1 \int_0^{1-x} (1-x-y)\,dy\,dx = 4\int_0^1 \left(y - xy - \frac{y^2}{2}\right)\Big|_0^{1-x} dx$$

$$= 4\int_0^1 1-x-x(1-x) - \frac{(1-x)^2}{2}\,dx = 4\int_0^1 1-x-x+x^2 - \frac{1}{2}(1-2x+x^2)\,dx$$

$$\begin{array}{r} 1-2x+x^2 \\ -\ \frac{1}{2} - x - \frac{x^2}{2} \\ \hline \end{array}$$

$$= 4\int_0^1 \left(\frac{1}{2} - x + \frac{x^2}{2}\right)dx = 4\left(\frac{x}{2} - \frac{x^2}{2} + \frac{x^3}{6}\Big|_0^1\right)$$

$$= 4\left[\tfrac{1}{2} - \tfrac{1}{2} + \tfrac{1}{6}\right] = 2/3.$$

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(2) Continued

$$M_{xy} = 4k\int_0^1 \int_0^{1-x} \frac{(1-x-y)^2}{2}\,dy\,dx = -4k\int_0^1 \frac{(1-x-y)^3}{6}\Big|_0^{1-x}$$

$$= -\frac{4}{6}k \int_0^1 \Big[\underbrace{(1-x-1+x)^3}_{=0} - (1-x)^3\Big]dx$$

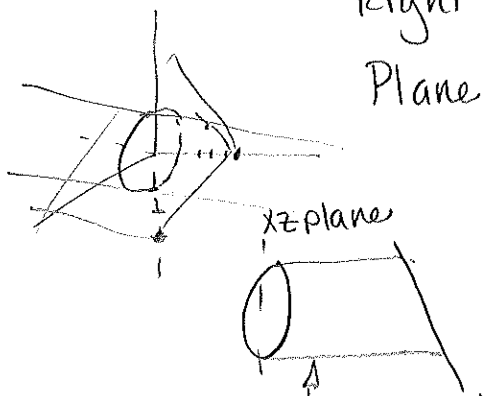
$$= +\frac{4}{6}k\left(\frac{-1}{4}\right)(1-x)^4\Big|_0^1 = \frac{-k}{6}(0-1) = \frac{k}{6}$$

$$\Rightarrow \bar{z} = \frac{k/6}{2/3} = 1/4$$ $\Rightarrow$ center of mass is $(\bar{x}, \bar{y}, \bar{z}) = (0, 0, 1/4)$

3)

Right cylinder $x^2+z^2=4$

Plane $x-y+z=-4$



Cylinder Plane $x-y+z=-4$.

$\rho(x,y,z)=ky$

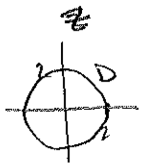
Inner most integral wrt y.

xz plane $= g_1(x,z)$

Plane $y = 4+x+z = g_2(x,z)$

$$M = \iint_D \int_0^{4+x+z} ky\,dy\,dA$$

What is D? D is the Bottom of our sideways "box". Specifically it is the circle $x^2+z^2=4$



Do integral in polar. $x=r\cos\theta$
$z=r\sin\theta$

$$M = \int_0^{2\pi}\int_0^2\int_0^{4-r(\cos\theta+\sin\theta)} ky \quad dy\,r\,dr\,d\theta$$

$$= k\int_0^{2\pi}\int_0^2 \frac{y^2}{2}\Big|_0^{4-r(\cos\theta+\sin\theta)} r\,dr\,d\theta$$

$$= \frac{k}{2}\int_0^{2\pi}\int_0^2 (4-r(\cos\theta+\sin\theta))^2\, r\,dr\,d\theta$$

$$= \frac{k}{2}\int_0^{2\pi}\int_0^2 \left[16r - 8r^2(\cos\theta+\sin\theta) + r^3(\cos\theta+\sin\theta)^2\right]dr\,d\theta$$

$$= \frac{k}{2}\int_0^{2\pi} 8r^2 - \frac{8r^3}{3}(\cos\theta+\sin\theta) + \frac{r^4}{4}(\cos\theta+\sin\theta)^2\Big|_0^2\, d\theta$$

$$= \frac{k}{2}\int_0^{2\pi} 32 - \frac{64}{3}(\cos\theta+\sin\theta) + \frac{64}{4}(\cos\theta+\sin\theta)^2\, d\theta$$

$$= k\,\frac{96\pi}{2}$$

$$V = \int_0^{2\pi} \int_0^{(\sqrt{3}-1/2)^{1/2}} \int_{r^2}^{\sqrt{4-r^2}} dz\, r\,dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^{(\sqrt{3}-1/2)^{1/2}} \left[r\sqrt{4-r^2} - r^3 \right] dr\, d\theta$$