Math 13 Worksheet #17: Surface integrals of vector fields

(1) True or false:

(a) If $F(x,y,z)$ is defined on an open region containing a smooth surface S, then $\int_S F(x,y,z)\cdot n dS$ measures the flow through the surface S in the direction n determined by the field F.

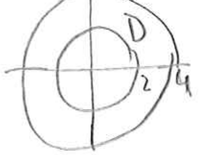
True

(b) In computing $\int_S F\cdot n dS$, the direction of the normal vector is irrelevant.

False

(c) In computing $\int_S F\cdot n dS$ with n pointing in the correct direction, we could use a scalar multiple of n, since the length will cancel in the dS term. True.

(2) Find the flux of $F(x,y,z) = < -xz, -yz, z^2 >$ through the surface S where S is the cone with equation $z = \sqrt{x^2+y^2}$ between $z = 2$ and $z = 4$ with n pointing outward.



Parameterize the surface $\bar{r}(x,y) = \langle x, y, \sqrt{x^2+y^2}\rangle$

$\bar{r}_x = \langle 1, 0, x(x^2+y^2)^{-1/2}\rangle$ $\quad \bar{r}_y = \langle 0, 1, y(x^2+y^2)^{-1/2}\rangle$

$$\bar{r}_x \times \bar{r}_y = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & 0 & x(x^2+y^2)^{-1/2} \\ 0 & 1 & y(x^2+y^2)^{-1/2} \end{vmatrix} = \langle -x(x^2+y^2)^{-1/2}, -y(x^2+y^2)^{-1/2}, 1\rangle$$

$\bar{F} = \langle -x\sqrt{x^2+y^2}, -y\sqrt{x^2+y^2}, x^2+y^2\rangle$

$\bar{F}\cdot(\bar{r}_x \times \bar{r}_y) = x^2+y^2 + x^2+y^2 = 2(x^2+y^2).$

$$\iint_S \bar{F}\cdot d\bar{S} = \iint_D \bar{F}\cdot(\bar{r}_x\times\bar{r}_y)dA = \iint_D 2(x^2+y^2)dA = \int_0^{2\pi}\int_2^4 2r^2 \, r dr d\theta$$

Integrate in polar
$2 \le r \le 4$
$0 \le \theta \le 2\pi$

$= 2\pi\left(\frac{2r^4}{4}\Big|_2^4\right)$

$= \pi(4^4 - 2^4)$

(3) Find the flux of $F(x,y,z) = < xz, 5z, y^2 >$ through the surface S where S is the region of the plane $12x - 9y + 3z = 20$, where $2 \le x \le 3$ and $5 \le y \le 10$, with n pointing upward.

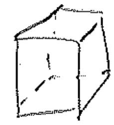
$z = \frac{10}{3} + 3y - 4x$ $\quad \bar{r}(x,y) = \langle x, y, \frac{10}{3} + 3y - 4x\rangle$

$\bar{r}_x = \langle 1, 0, -4\rangle$ $\quad \bar{r}_y = \langle 0, 1, 3\rangle$

$$\bar{r}_x \times \bar{r}_y = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & 0 & -4 \\ 0 & 1 & 3 \end{vmatrix} = \langle 4, -3, 1\rangle$$

$$\bar{F}\cdot(\bar{r}_x\times\bar{r}_y) = 4xz - 15z + y^2 = \frac{40x}{3} + 12xy + 16x^2 - \frac{150}{3} - 45y - 60x + y^2$$

$$\iint_S \bar{F}\cdot(\bar{r}_x\times\bar{r}_y)dA = \int_2^3\int_5^{10} \bar{F}\cdot(\bar{r}_x\times\bar{r}_y)\,dy\,dx =$$



(4) Suppose an electric field is given by $\boldsymbol{E}(x,y,z) = < 2y, 2xy, yz >$. Compute the flux $\int_S \boldsymbol{E} \cdot \boldsymbol{n} dA$ of the field through the unit cube $[0,1] \times [0,1] \times [0,1]$.

$$\int_S \bar{E}\cdot\bar{n}\,dA = \int_{\text{Face1}} \bar{E}\cdot\bar{n}\,dA + \int_{\text{Face2}} \bar{E}\cdot\bar{n}\,dA \;.\;.\; + \int_{\text{Face6}} \bar{E}\cdot\bar{n}\,dA$$

where

Face1: $\{(x,y,0) : (x,y)\in[0,1]^2\}$ $\quad \bar{n} = \langle 0,0,-1\rangle$

Face2: $\{(x,y,1) : (x,y)\in[0,1]^2\}$ $\quad \bar{n} = \langle 0,0,1\rangle$

Face3: $\{(0,y,z) : (y,z)\in[0,1]^2\}$ $\quad n = \langle -1,0,0\rangle$

Face4: $\{(1,y,z) : (y,z)\in[0,1]^2\}$ $\quad n = \langle 1,0,0\rangle$

Face5: $\{(x,0,z) : (x,z)\in[0,1]^2\}$ $\quad n = \langle 0,-1,0\rangle$

Face6: $\{(x,1,z) : (x,z)\in[0,1]^2\}$ $\quad n = \langle 0,1,0\rangle$

$$\iint_{\text{Face1}} \bar{E}\cdot\bar{n}\,dA = \int_0^1\int_0^1 \langle 2y, 2xy, 0\rangle\cdot\langle 0,0,-1\rangle\,dA = 0.$$

$$\iint_{\text{Face2}} \bar{E}\cdot\bar{n}\,dA = \int_0^1\int_0^1 \langle 2y, 2xy, y\rangle\cdot\langle 0,0,1\rangle\,dA = \int_0^1\int_0^1 y\,dy\,dx = \frac{1}{2}$$

$$\iint_{\text{Face3}} \bar{E}\cdot\bar{n}\,dA = \iint_0^1 \langle 2y, 0, yz\rangle\cdot\langle -1,0,0\rangle\,dy\,d\cancel{x}z = \int_0^1\int_0^1 -2y\,dy\,dz = \int_0^1 -y^2\Big|_0^1 dz = -1$$

$$\iint_{\text{Face4}} \bar{E}\cdot\bar{n}\,dA = \int_0^1\int_0^1 \langle 2y, 2y, yz\rangle\cdot\langle 1,0,0\rangle\,dy\,dz = 1$$

$$\iint_{\text{Face5}} \bar{E}\cdot\bar{n}\,dA = \int_0^1\int_0^1 \langle 0,0,0\rangle\cdot\bar{n}\,dA = 0$$

$$\iint_{\text{Face6}} \bar{E}\cdot\bar{n}\,dA = \int_0^1\int_0^1 \langle 2, 2x, z\rangle\cdot\langle 0,1,0\rangle\,dx\,dz = \int_0^1\int_0^1 2x\,dx\,dz = 1$$

$$\iint_S \bar{E}\cdot\bar{n}\,dA = 0 + 1/2 - 1 + 1 + 0 + 1 = 1 + 1/2 = 3/2$$