Math 13, Spring 2014 – Homework Solutions Week 6

(1) (Problem #2, Chapter 16.4) Evaluate the line integral below by using two methods: Direct evaluation and Green's Theorem. Verify that the answers are identical.

$$\int_C xy\,dx + x^3\,dy$$

where C is the rectangle (with positive orientation) with vertices (0, 0), (3, 0), (3, 1) and (0, 1).

Solution. Using Green's Theorem with $P = xy$ and $Q = x^3$, we note that both P an Q are have continuous partial derivatives on the plane, so Green's Theorem applies.

$$\begin{aligned}\iint_R 3x^2 - x\,dA &= \int_0^3 \int_0^1 3x^2 - x\,dy\,dx \\ &= (3x^2 - x)y\Big|_0^1 \\ &= \int_0^3 3x^2 - x\,dx \\ &= x^3 - \frac{x^2}{2}\Big|_0^3 \\ &= \frac{45}{2}\end{aligned}$$

To compute the integral directly, we consider C to be the union of 4 line segments.

On $(0,0) \to (3,0)$: Use x as the parameter; $y = 0$ and $dy = 0$

$$\int_0^3 0\,dx = 0$$

On $(3,0) \to (3,1)$: Use y as the parameter; $x = 3$ and $dx = 0$

$$\int_0^1 3^3\,dy = 27$$

On $(3,1) \to (0,1)$: Use x as the parameter; $y = 3$, $dy = 0$

$$\int_3^0 x\,dx = \left.\frac{x^2}{2}\right|_3^0 = -\frac{9}{2}$$

On $(0,1) \to (0,0)$: Use y as the parameter; $x = 0$, $dx = 0$

$$\int_1^0 0\,dy = 0$$

Value of integral $= 0 + 27 - \frac{9}{2} + 0 = \frac{45}{2}$

(2) (Problem #22, Chapter 16.4) Let D be a region bounded by a simple closed path C in the xy-plane. Use Green's Theorem to prove that the coordinates $(\bar{x}, \bar{y})$ of the centroid (the centroid is the center of mass of D, if we assume that D is a lamina of uniform density) of D are

$$\bar{x} = \frac{1}{2A}\int_C x^2\,dy, \quad \bar{y} = -\frac{1}{2A}\int_C y^2\,dx$$

Solution. We may assume the density equals 1 in our calculations, since because it is constant it will cancel when we compute $\bar{x}$ and $\bar{y}$. The mass then is

$$\iint_D 1\,dA = A$$

The calculations for M_y and M_x are

$$\begin{aligned} M_y &= \iint_D x\,dA \\ &= \int_C \frac{x^2}{2}\,dy \quad \left(\text{using Green's Theorem with } Q = \frac{x^2}{2},\ P = 0\right) \\ M_x &= \iint_D y\,dA \\ &= \int_C -\frac{y^2}{2}\,dx \quad \left(\text{using Green's Theorem with } P = -\frac{y^2}{2},\ Q = 0\right) \end{aligned}$$

Therefore $\bar{x} = \dfrac{\int_C \frac{x^2}{2}\,dy}{A}$ and $\bar{y} = \dfrac{\int_C -\frac{y^2}{2}\,dx}{A}$ as desired.

(3) (Problem #12, Chapter 16.5) Let f be a scalar function and $F : \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field. State whether each expression below is meaningful. If not, explain why. If so, state whether it is a scalar function or a vector field, and give a brief explanation why.

(a) curl f

(b) grad f

(c) div F

(d) curl grad f

(e) grad F

(f) grad div F

(g) div grad f

(h) grad div f

(i) curl curl F

(j) div div F

(k) grad f $\times$ div F

(l) div curl grad f

Solution.

(a) No – defined only for vector field

(b) Yes – result is a vector field

(c) Yes – result is a scalar function

(d) Yes – result is the zero vector field

(e) No – defined only for scalar function

(f) Yes – result is a vector field

(g) Yes – result is a scalar function

(h) Yes – result is a vector field

(i) Yes – result is a vector field

(j) No – div F is a scalar function, second div operator is not defined

(k) No – div F is not a vector

(l) Yes – result is a scalar function

(4) (Problem #20, Chapter 16.5) Is there a smooth vector field G on $\mathbb{R}^3$ such that $\nabla \times G = \langle xyz, y^2z, yz^2 \rangle$? Explain.

Solution.

If G is a smooth vector field, we must have div curl $G = 0$. But

$$\text{div } \langle xyz, y^2z, yz^2 \rangle = yz + 2yz + 2yz = 5yz \neq 0$$

so such a G does not exist.