Homework #5

1) $\int_C (x+y)\,dx + (x+z)\,dy + (x+y)\,dz = I$

C: line from C_1 (0,0,0) to (1,0,1) & line segment from C_2 (1,0,1) to (0,1,2)

$C_1: \bar{r}_1(t) = (1-t)\langle 0,0,0\rangle + t\langle 1,0,1\rangle \quad 0 \le t \le 1$

$C_2: \bar{r}_2(t) = (1-t)\langle 1,0,1\rangle + t\langle 0,1,2\rangle \quad 0 \le t \le 1$

$= \langle 1-t,\ t,\ 1-t+2t\rangle = \langle 1-t,\ t,\ 1+t\rangle$

$I_1 = \int_{C_1} (x+y)\,dx + (x+z)\,dy + (x+y)\,dz =$

$= \int_0^1 [(x(t)+y(t))\,x'(t) + (x(t)+z(t))\,y'(t) + (x(t)+y(t))\,z'(t)]\,dt$

$= \int_0^1 [\ t\cdot 1 + (2t)\cdot 0 + t\cdot 1]\,dt = \int_0^1 2t\,dt = t^2\Big|_0^1 = 1$

$I_2 = \int_{C_2} (x+y)\,dx + (x+z)\,dy + (x+y)\,dt = \int_0^1 [(1-t+t)(-1) + (1-t+1+t)\cdot 1 + (1-t+t)\cdot 1]\,dt$

$= \int_0^1 2-2t\,dt = 2t-t^2\Big|_0^1 = 1$

$I = I_1 + I_2 = 2$

$\bar{r}(t) = (1-t)\langle 2,0,0\rangle + t\langle 2,1,5\rangle \quad 0 \le t \le 1$

$= \langle 2, t, 5t\rangle$

2) $W = \int_C \bar{F}\cdot d\bar{r} = \int_0^1 \bar{F}\cdot\bar{r}'(t)\, dt$

$= \int_0^1 \frac{\bar{r}\cdot\bar{r}'}{|\bar{r}|^3}\, dt = \int_0^1 \frac{\langle 2,t,5t\rangle\cdot\langle 0,1,5\rangle}{(4+26t^2)^{3/2}}\, dt$

$= \int_0^1 \frac{26t}{(4+26t^2)^{3/2}}\, dt \qquad u = 4+26t^2$

$du = 2(26)t\, dt$

$= \frac{1}{2}\int_4^{30} \frac{1}{u^{3/2}}\, du = \frac{1}{2}\frac{-2}{1}u^{-1/2}\Big|_4^{30} = -\frac{1}{\sqrt{30}} + \frac{1}{2}$

3) $\int_C \bar{B}\cdot d\bar{r} = \mu_0 I$ ①

$\bar{r}(t) = \langle r\cos t, r\sin t, 0\rangle \quad 0 \le t \le 2\pi$

Since $\bar{B}$ is tangent to the circle, $\bar{B} = k\langle -y, x, 0\rangle$

k: some constant

$\int_C \bar{B}\cdot d\bar{r} = k\int_0^1 \langle -r\sin t, r\cos t, 0\rangle\cdot\langle -r\sin t, r\cos t, 0\rangle dt$

$= k\int_0^{2\pi} r^2\, dt = k2\pi r^2$

By ① $\quad 2\pi k r^2 = \mu_0 I \Rightarrow k = \frac{\mu_0 I}{2\pi r^2}$

Note $|\bar{B}| = kr = B$

$\Rightarrow B = \frac{\mu_0 I}{2\pi r}$

4) $\bar{F} = \langle P, Q, R \rangle$

If $\bar{F}$ is conservative $\Rightarrow \exists$ a function $f(x,y,z)$. st $\nabla f = \bar{F}$

$\Rightarrow$ $P_y = f_{xy} = f_{yx} = Q_x$ by Clairaut's Thm.

likewise the others are true.

5) $\bar{F}(x,y) = \langle (1+xy)e^{xy},\ x^2 e^{xy} \rangle$

Integrate the 2nd term

$f(x,y) = \int x^2 e^{xy}\, dy = xe^{xy} + C_1(x)$ where $C_1(x)$ is

a function purely of x.

$\frac{\partial f}{\partial x} = (1+xy)e^{xy} + C_1'(x) = (1+xy)e^{xy}$

$C_1'(x) = 0. \rightarrow C_1 = k$

$f(x,y) = xe^{xy} + k$

$\bar{r}(t) = \langle \cos t, 2\sin t \rangle \qquad 0 \le t \le \pi/2$

We get to use the Fundamental Thm of Calc.

$$\int_C \bar{F} \cdot d\bar{r} = \int_C \nabla f \cdot d\bar{r} = f(\bar{r}(\pi/2)) - f(\bar{r}(0))$$

$$= f((0,2)) - f((1,0))$$

$$= 0 - 1 = -1$$

$\bar{r} = \langle x, y, z \rangle \qquad |\bar{r}| = (x^2+y^2+z^2)^{3/2}$

6) $F(\bar{r}) = \dfrac{c\bar{r}}{|\bar{r}|^3}$

We could try to calculate work W directly but we should use the Fundamental Thm of Calc especially since we don't know the curve.

The potential function

$$f(x,y,z) = \frac{-c}{\sqrt{x^2+y^2+z^2}}$$

So

$$W = \int_C \bar{F} \cdot d\bar{r} = f(P_2) - f(P_1)$$

$$= \frac{-c}{d_2} + \frac{c}{d_1}$$

where d_1 = distance of P_1 from the origin

d_2 = distance of P_2 from the origin.

Notice that the curve cannot go through the origin.