Math 13, Spring 2014 – Homework Solutions Week 4

(1) (Problem #66, Chapter 12.5) Find parametric equations for the line through the point $P(0, 1, 2)$ that is perpendicular to the line L given by equations $x = 1 + t$, $y = 1 - t$, $z = 2t$ and intersects this line.

Solution. Let $Q = (1 + a, 1 - a, 2a)$ be the point on L such that vector $\overrightarrow{PQ}$ is perpendicular to L. $\overrightarrow{PQ} = \langle 1 + a, -a, 2a - 2\rangle$. The vector giving the direction of L is $\vec{v} = \langle 1, -1, 2\rangle$.
$\overrightarrow{PQ} \cdot \vec{v} = 1 + a + a + 4a - 4 = 0$, so $a = \frac{1}{2}$.
Therefore $\overrightarrow{PQ} = \left\langle -\frac{3}{2}, -\frac{1}{2}, 0\right\rangle$ and the parametric equations of the solution line are $x = \frac{3}{2}s$, $y = 1 - \frac{1}{2}s$, $z = 2$.

(2) (Problem #50, Chapter 12.3) A tow truck drags a stalled car along a road. The chain makes an angle of 30 degrees with the road and tension in the chain is 1500 N. How much work is done by the truck in pulling the car 1 km?

Solution. (1500 N) (1 km) $(\cos(30^o)) = 750\sqrt{3}$ Newton-kilometers

(3) (Problem #48, Chapter 12.3) Suppose that $\vec{a}$ and $\vec{b}$ are nonzero.

(a) Under what circumstances is $\text{comp}_{\vec{a}}\vec{b} = \text{comp}_{\vec{b}}\vec{a}$?

Solution. $\dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}|} = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$ means $|\vec{a}| = |\vec{b}|$ (the vectors have the same magnitude).

(b) Under what circumstances is $\text{proj}_{\vec{a}}\vec{b} = \text{proj}_{\vec{b}}\vec{a}$?

Solution. $\dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2}\vec{a} = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2}\vec{b}$ means $|\vec{a}|\frac{\vec{b}}{|\vec{b}|} = |\vec{b}|\frac{\vec{a}}{|\vec{a}|}$, or $\vec{a} = \frac{|\vec{a}|^2}{|\vec{b}|^2}\vec{b}$.
$|\vec{a}|\frac{\vec{b}}{|\vec{b}|} = |\vec{b}|\frac{\vec{a}}{|\vec{a}|}$ implies the vectors have the same magnitude, while $\vec{a} = \frac{|\vec{a}|^2}{|\vec{b}|^2}\vec{b}$ implies they point in the same direction. Therefore $\vec{a}$ equals $\vec{b}$.

(4) (Problem #72, Chapter 14.3) If $g(x, y, z) = \sqrt{1 + xz} + \sqrt{1 - xy}$, find g_{xyz}.

Solution. Since $\frac{\partial}{\partial y}\sqrt{1+xz} = 0$ and $\frac{\partial}{\partial z}\sqrt{1-xy} = 0$, $g_{xyz} = 0$.

(5) Suppose you need to know an equation for the tangent plane to a surface S at the point $P(1,2,3)$. You do not have an equation for S but you know that the curves $r_1(t) = \langle 1+3t, 2-t^2, 3-4t-t^2\rangle$, $r_2(u) = \langle 1+u^2, 2u^3+2, 2u+3\rangle$ both lie on S. Find an equation of the tangent plane at P.

Solution. P is on both curves, where $t = 0$ and $u = 0$. The vectors $r_1'(0)$ and $r_2'(0)$ will be tangent to the surface, and therefore their cross product will be perpendicular to S at P and we can use it in the equation of the tangent plane.
$r_1'(t) = \langle 3, -2t, -4-2t\rangle$; $r_1'(0) = \langle 3, 0, -4\rangle$
$r_2'(u) = \langle 2u, 6u^2, 2\rangle$; $r_2'(u) = \langle 0, 0, 2\rangle$
$\langle 3,0,-4\rangle \times \langle 0,0,2\rangle = \langle 0,-6,0\rangle$ so the equation of the tangent plane is $-6(y-2) = 0$.

(6) (Problem #62, Chapter 14.6) Show that the pyramids cut off from the first octant by any tangent planes to the surface $xyz = 1$ at points in the first octant must all have the same volume.

Solution. Let $P(a,b,c)$ be a point on the surface in the first octant. To find the equation of the tangent plane, we need to evaluate the partial derivatives at P.
$f_x = yz$; $f_x(P) = bc$
$f_y = xz$; $f_y(P) = ac$
$f_z = xy$; $f_z(P) = ab$
The tangent plane is $bc(x-a) + ac(y-b) + ab(z-c) = 0$, which can be rearranged using the fact that $abc = 1$ to get $x/a + y/b + z/c = 3$. The intercepts of the plane are the points $(3a,0,0)$, $(0,3b,0)$, $(0,0,3c)$, and these along with $(0,0,0)$ determine the pyramid.
The base of the pyramid is the triangle with vertices $(0,0,0)$, $(3a,0,0)$, $(0,3b,0)$ which has area $\frac{9}{2}ab$. The height of the pyramid is $3c$, so its volume is $\frac{1}{3}\cdot\frac{9}{2}ab\cdot 3c = \frac{9}{2}abc = \frac{9}{2}$. Since (a,b,c) is any point on the surface in the first octant, all pyramids have the same volume.