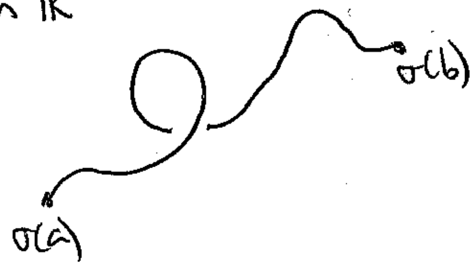
Scalar line integrals:

$\sigma : [a,b] \to \mathbb{R}^3 \quad f : \mathbb{R}^3 \to \mathbb{R}$

Motivating example: Think of ~~$\sigma[a,b]$~~ $\sigma(t)$ as a piece of wire sitting in $\mathbb{R}^3$



Let ~~$f(x,y,z)$~~ $f(\sigma(t))$ be the charge ~~at~~ density of the wire at the point $\sigma(t)$. Then total charge of the wire is given by

$$\underset{\text{the wire}}{\sum} \underbrace{f(\sigma(t))}_{\text{charge density on the wire}} \underbrace{\|\Delta\sigma\|}_{\text{tiny length of wire}} . \quad \text{As } \Delta\sigma \to 0,$$

$$\Downarrow$$

$$\int_a^b f(\sigma(t)) \, \|\sigma'(t)\| \, dt.$$

Def. The scalar line integral of along σ is given by

$$\int_\sigma f \, ds = \int_a^b f(\sigma(t)) \|\sigma'(t)\| \, dt.$$

e.g. $\underset{1 \le t \le 3}{\sigma(t) =} (t, t, t^{3/2})$, $f(x,y,z) = \dfrac{x+z}{y+z}$. Find $\int_\sigma f \, ds$.

$f(\sigma(t)) = \dfrac{t + t^{3/2}}{t + t^{3/2}}$. $\|\sigma'(t)\| = \| (1, 1, \tfrac{3}{2}\sqrt{t}) \|$

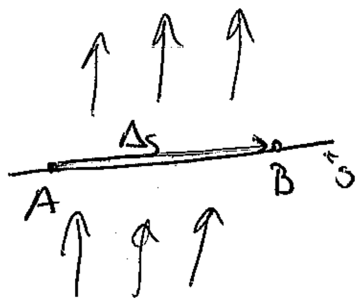
$= \sqrt{2 + \tfrac{9}{4} t}$

$\int_1^3 \cancel{\phi}\, 1 \sqrt{(2 + \tfrac{3}{4} t)} \, dt = \cancel{(2t + \tfrac{3}{8} t^{3/2})}\Big|_{t=1}^{t=3}$

$= \dfrac{4}{9} \cdot \dfrac{2}{3} \left(2 + \dfrac{9}{4} t\right)^{3/2} \Bigg|_{t=1}^{t=3} = \dfrac{1}{27} \left(35^{3/2} - 17^{3/2}\right).$

Vector line integrals

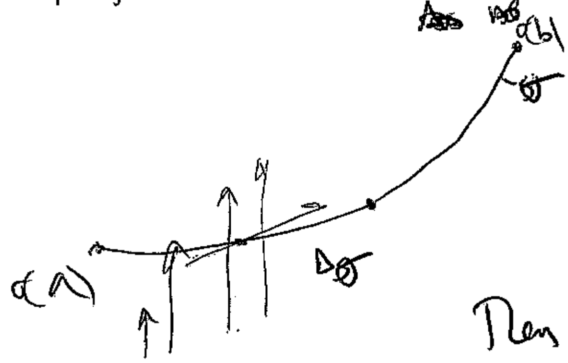
Let F be a constant vector field, and suppose we want to push a particle in a straight line from A to B



The work done ~~to~~ to move it is

$$\text{Work} = F \cdot \Delta s$$

In general F need not be constant and s need not be straight. But if Δs is small enough we can pretend its straight and if Δs is small enough we can ~~pretend~~ pretend F is pretty much constant.



Then the total work is given by

$$\underset{\text{along the path}}{\sum} F \cdot \Delta\sigma = \int_a^b \underset{\substack{\uparrow \\ \text{vector field} \\ \text{at the point}}}{F(\sigma(t))} \cdot \underset{\substack{\uparrow \\ \text{tiny piece} \\ \text{of the "linearized"} \\ \text{path}}}{\sigma'(t)}\, dt$$

Defn: The vector line integral of F along $\sigma : [a,b] \to \mathbb{R}^3$ is given by

$$\int_\sigma F \cdot ds = \int_a^b F(\sigma(t)) \cdot \sigma'(t)\, dt.$$

eg $F(x,y,z) = (x,y,z)$, $\sigma(t) = (2t+1, t, 3t-1)$ $\quad 0 \le t \le 1$

$$\int_\sigma F \cdot ds = \int_0^1 (2t+1, t, 3t-1) \cdot (2,1,3)\, dt$$

$$= \int_0^1 4t+2+t+3t-3\, dt = \int_0^1 8t-1\, dt$$

$$= 4t^2 - t \Big|_0^1 = 3$$

Other ways of thinking about vector line integrals:

Circulations

Recall we defined the unit tangent vector $T(t)$ to be

$$T(t) = \frac{\sigma'(t)}{\|\sigma'(t)\|}.$$

Then

$$\int_\sigma F \cdot ds = \int_a^b F(\sigma(t)) \cdot \sigma'(t)\, dt$$

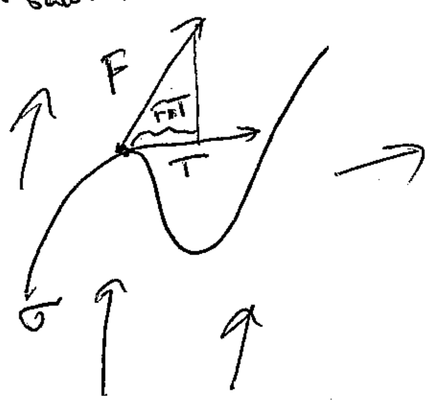
vector line integral

$$= \int_a^b F(\sigma(t)) \cdot \frac{\sigma'(t)}{\|\sigma'(t)\|} \|\sigma'(t)\|\, dt$$

$$= \int_a^b F(\sigma(t)) \cdot T(t)\, \|\sigma'(t)\|\, dt$$

$$= \int_\sigma (F \cdot T)\, ds$$

scalar line integral

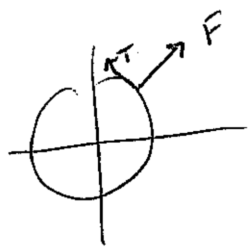


~~Eg. σ(t) = (3cost, 3sint)~~

$$\boxed{\int_\sigma F \cdot ds \text{ represents the line integral of the tangential component along the path } \sigma.}$$

when σ is a closed path, i.e., when $\sigma(a) = \sigma(b)$, $\int_\sigma F \cdot ds$ is called the circulation of F along σ.

e.g. $\sigma = (3\cos t, 3\sin t)$, $F(x,y) = (x,y)$. What is $\int_\sigma F \cdot ds$ without integrating?



F has no tangential component. So $F \cdot T = 0$. So $\int_\sigma F \cdot ds = 0$.