Def. A joint probability density function for two independent variables x and y, is a function $f(x,y)$ s.t

(i) $f(x,y) \geq 0$

(ii) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$

eg $\frac{1}{\pi} e^{-x^2-y^2}$ is such a thing.

eg If a, b are fixed >0. What value of C (in terms of a and b) will make

$$f(x,y) = C e^{-a|x|-b|y|} \text{ a pdf?}$$

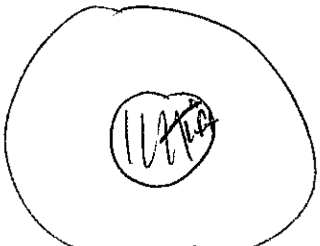
Soln: $C = \frac{ab}{4}$

$$C\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 4\int_0^\infty\int_0^\infty C e^{-ax-by}\,dx\,dy$$

$$= 4C\int_0^\infty \frac{1}{a} e^{-by}\,dy = \frac{4C}{ab}.$$

If $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$ then $C = \frac{ab}{4}$

eg If $f(x,y) = \frac{1}{\pi} e^{-x^2-y^2}$ ~~and~~ is the distribution of randomly shot arrows shooting at a bullseye of radius 1. What's the probability of hitting the bullseye?



If (x,y) chosen randomly, $\text{Prob}((x,y) \in D) = \iint_D f(x,y)\,dA$

$$\frac{1}{\pi}\int_0^{2\pi}\int_0^1 e^{-r^2} r\,dr\,d\theta = \frac{1}{\pi}\int_0^{2\pi} -\frac{1}{2}e^{-r^2}\Big|_0^1$$

$$= \frac{1}{\pi}\int_0^{2\pi} -\frac{1}{2e} - -\frac{1}{2}e^0$$

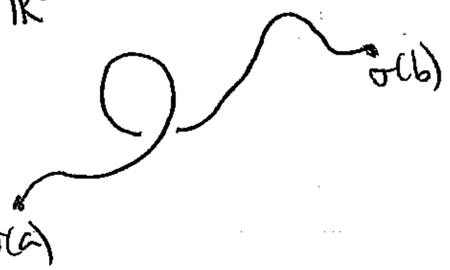
$$= \frac{1}{\pi}\int_0^{2\pi} \frac{1}{2} - \frac{1}{2e} = 2\left(\frac{1}{2} - \frac{1}{2e}\right)$$

$$= \boxed{1 - \frac{1}{e}}$$

Scalar line integrals:

$\sigma: [a,b] \to \mathbb{R}^3 \quad f: \mathbb{R}^3 \to \mathbb{R}$

Motivating example: Think of ~~$\sigma[a,b]$~~ $\sigma(t)$ as a piece of wire sitting in $\mathbb{R}^3$



Let ~~$f(x,y,z)$~~ $f(\sigma(t))$ be the charge ~~at~~ density of the wire at the point $\sigma(t)$. Then total charge of the wire is given by

$$\underbrace{\sum}_{\text{the wire}} \underbrace{f(\sigma(t))}_{\text{charge density on the wire}} \underbrace{\|\Delta\sigma\|}_{\text{tiny length of wire}}$$

As $\Delta\sigma \to 0$,

$$\Downarrow$$

$$\int_a^b f(\sigma(t))\, \|\sigma'(t)\|\, dt.$$

Def. The scalar line integral of f along σ is given by

$$\int_\sigma f\, ds = \int_a^b f(\sigma(t))\, \|\sigma'(t)\|\, dt.$$

e.g. $\sigma(t) = (t, t, t^{3/2})$, $1 \le t \le 3$, $f(x,y,z) = \dfrac{x+z}{y+z}$. Find $\int_\sigma f\, ds$.

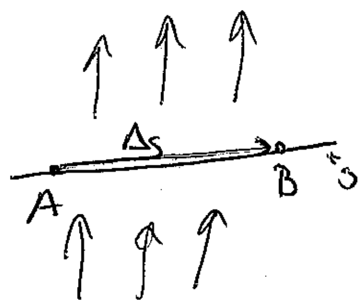
$f(\sigma(t)) = \dfrac{t+t^{3/2}}{t+t^{3/2}}$. $\quad \|\sigma'(t)\| = \|(1, 1, \tfrac{3}{2}\sqrt{t})\| = \sqrt{2 + \tfrac{9}{4}t}$

$$\int_1^3 1 \sqrt{(2+\tfrac{3}{4}t)}\, dt = \cancel{\left(2t+\tfrac{3}{8}t^{3/2}\right)\Big|_{t=1}^{t=3}} = \frac{4}{9}\cdot\frac{2}{3}\left(2+\frac{9}{4}t\right)^{3/2}\Bigg|_{t=1}^{t=3} = \frac{1}{27}\left(35^{3/2} - 17^{3/2}\right).$$

Vector line integrals

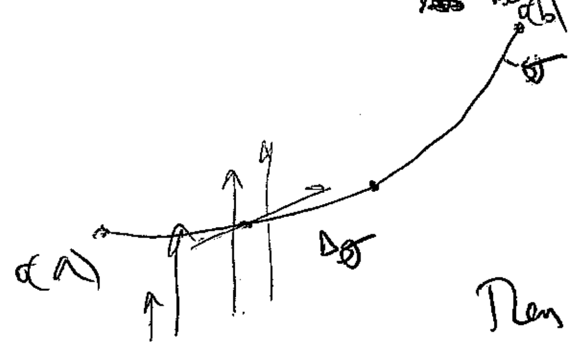
Let F be a constant vector field, and suppose we want to push a particle in a straight line from A to B

The work done to move it is



Work $= F \cdot \Delta s$

In general F need not be constant and s need not be straight. But if Δs is small enough we can pretend its straight and if Δs is small enough we can pretend F is pretty much constant.



Then the total work is given by

$$\sum F\cdot\Delta\sigma = \int_a^b F(\sigma(t))\cdot \sigma'(t)\,dt$$

along the path

vector field at the point

tiny piece of the "linearized" path

<u>Defn</u>: The vector line integral of F along $\sigma : [a,b] \to \mathbb{R}^3$ is given by

$$\int_\sigma F\cdot ds = \int_a^b F(\sigma(t))\cdot \sigma'(t)\,dt.$$

eg $F(x,y,z) = (x,y,z)$, $\sigma(t) = (2t+1, t, 3t-1)$ $\quad 0 \le t \le 1$

$$\int_\sigma F\cdot ds = \int_0^1 (2t+1, t, 3t-1)\cdot(2,1,3)\,dt$$

$$= \int_0^1 4t+2+t+3t-3\,dt = \int_0^1 8t-1\,dt$$

$$= 4t^2 - t\Big|_0^1 = 3$$

Other ways of thinking about vector line integrals:

Circulations

Recall we defined the unit tangent vector $T(t)$ to be

$$T(t) = \frac{\sigma'(t)}{\|\sigma'(t)\|}.$$

Then

$$\int_\sigma F\cdot ds = \int_a^b F(\sigma(t))\cdot \sigma'(t)\,dt$$

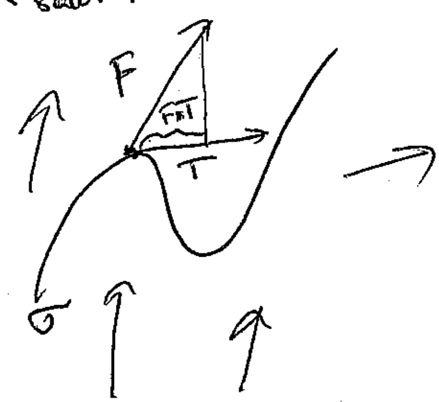
vector line integral

$$= \int_a^b F(\sigma(t))\cdot \frac{\sigma'(t)}{\|\sigma'(t)\|}\|\sigma'(t)\|\,dt$$

$$= \int_a^b F(\sigma(t))\cdot T(t)\,\|\sigma'(t)\|\,dt$$

$$= \int_\sigma (F\cdot T)\,ds$$

scalar line integral

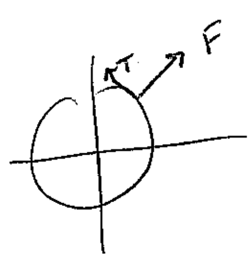


~~Eg. σ(t) = (3cos t, 3sin t)~~

So $\int_\sigma F\cdot ds$ represents the line integral of the tangential component along the path σ.

When σ is a closed path, i.e., when $\sigma(a)=\sigma(b)$, $\int_\sigma F\cdot d\vec{s}$ is called the circulation of F along σ.

e.g. $\sigma = (3\cos t, 3\sin t)$ $F(x,y) = (x,y)$ What is $\int_\sigma F\cdot ds$ without integrating?



F has no tangential component. So $F\cdot T = 0$. So $\int_\sigma F\cdot ds = 0$.

Differential form:

$\sigma(t) = (x(t), y(t), z(t)) : [a,b] \to \mathbb{R}^3$ a path

$F(x,y,z) = (M(x,y,z), N(x,y,z), P(x,y,z))$ a vector field

Then

$$\int_\sigma F \cdot ds = \int_a^b (M(x,y,z) \cdot x'(t) + N(x,y,z) \cdot y'(t) + P(x,y,z) z'(t))\, dt$$

Notice $x'(t)\,dt = dx$
$y'(t)\,dt = dy$ so
$z'(t)\,dt = dz$

$$= \int_a^b M(x,y,z)\, dx + N(x,y,z)\, dy + P(x,y,z)\, dz$$

e.g We compute $\int_\sigma (y+z)dx + (x+z)dy + (x+y)\,dz$

where $\sigma(t) = (t, t^2, t^3)$. $0 \leq t \leq 1$

Sol'n: $\int_0^1 (t^2+t^3)\,dt + (t+t^3)2t\,dt + (t+t^2)3t^2\,dt = 3.$

What happens if you reparameterize?

$\sigma(t) : [0, 2\pi) \to \mathbb{R}^2$ $(\cos t, \sin t)$ — a circle traced out counter clockwise at 1 radian/sec

vs

$\varphi(t) = [0, \pi] \to \mathbb{R}^2$ $(\cos 2t, \sin 2t)$ — a circle traced out counter clockwise at 2 radian/sec

vs

$\psi(t) = [0, 2\pi] \to \mathbb{R}^2$ $(-\cos t, -\sin t)$ — a circle traced out clockwise at 1 radian/sec