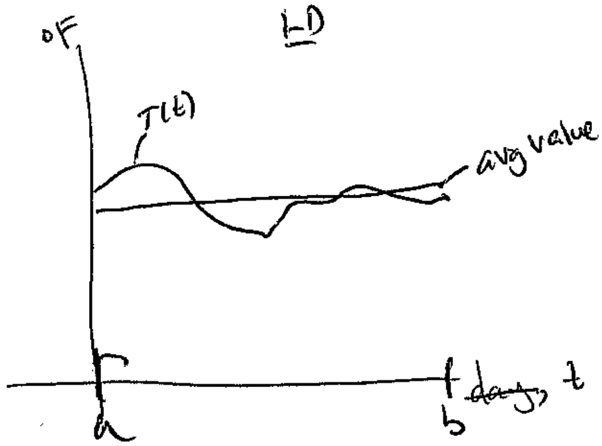
Average values of functions:

1-D



How do we define the average value?

Area under the ~~elapsed tim~~ avg value over the elapsed time should equal the area under T(t) over the elapsed time

i.e.

$$\int_a^b \text{avgvalue}\, dt = \int_a^b T(t)\,dt$$

$$\Downarrow$$

$$\text{avgvalue}(b-a) = \int_a^b T(t)\,dt$$

$$\text{avgvalue} = \frac{1}{b-a}\int_a^b T(t)\,dt.$$

Def: Let $f:[a,b]\to\mathbb{R}$ be an integrable function of one variable.

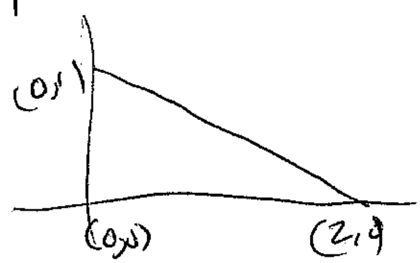
We define the average (mean) value of f to be

$$[f]_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Note

$$[f]_{avg} = \frac{\int_a^b f(x)\,dx}{\int_a^b 1\,dx}$$

2-D Suppose that the thickness of a triangular metal plate at (x,y) is given by $(1+xy)$. Find the average thickness.



$z = f(x,y) = 1+xy$ is a surface.

Again we want average of $f(x,y)$ to be so that

volume under average = volume under $f(x,y)$

$$\iint_D \text{avg}\, dA = \iint_D f(x,y)\,dA.$$

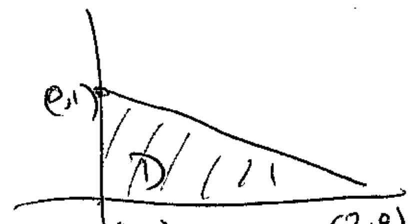
i.e.,

$$avg = \frac{\iint_D f(x,y)\,dA}{\iint_D 1\,dA}.$$

<u>Definition</u>: Let $f: D \subseteq \mathbb{R}^2 \to \mathbb{R}$ be an integrable function. The average value of f on D is

$$[f]_{avg} = \frac{\iint_D f(x,y)\,dA}{\iint_D 1\,dA} = \frac{\iint_D f(x,y)\,dA}{\cancel{\iint} \text{ area of } D}$$

<u>Back to metal plate problem</u>:



$D =$ ~~$0 \le x \le 2,\ 0 \le y \le 1 - \frac{1}{2}x$~~

$0 \le 1 \le y \quad 0 \le x \le 2 - 2y$

$$\iint_D \overset{\text{thickness}}{1+xy}\,dA = \int_0^1 \int_0^{2-2y} 1+xy\,dx\,dy$$

$$= \int_0^1 \left(x + \frac{x^2 y}{2}\right)_0^{2-2y} dy$$

$$= \int_0^1 (2-2y) + \frac{y(2-2y)^2}{2}\,dy$$

$$= \frac{7}{6}.$$

area of D = area of triangle with base 2 and height 1 = 1.

~~aver~~ $[f]_{avg} = \frac{7}{6}$

Def: Let $f: W \subseteq \mathbb{R}^3 \to \mathbb{R}$ be integrable. Then

$$[f]_{avg} = \frac{\iiint_W f \, dV}{\iiint_W dV} = \frac{\iiint_W f \, dV}{\text{volume } V}$$

e.g Earlier example revisited.

Let W be the region bounded by ~~f(x,y)=~~ $z = 9 - x^2 - y^2$ and $z = 3x^2 + 3y^2 - 16$. Suppose the temperature in W is given by $T(x,y,z) = z(x^2+y^2)$.

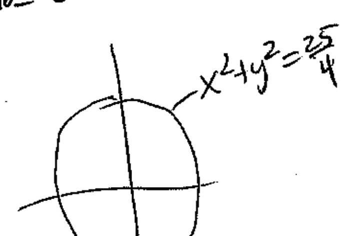
We set up last week the following integral:

$$\iiint_W dV = \int_{-\frac{5}{2}}^{\frac{5}{2}} \int_{-\sqrt{\frac{25}{4}-y^2}}^{\sqrt{\frac{25}{4}-y^2}} \int_{3x^2+3y^2-16}^{9-x^2-y^2} dz\,dx\,dy.$$

and noted this would be hard to compute. Let's do a change of variables to cylindrical and see that its easier. dV in cylindrical is $r\,dr\,d\theta\,dz$

Height: $3x^2+3y^2-16 \le z \le 9-x^2-y^2$ $\qquad r^2 = x^2+y^2 \Rightarrow 3r^2-16 \le z \le 9-r^2$

Width & length:



$0 \le \theta \le 2\pi$

$0 \le r \le \frac{5}{2}$

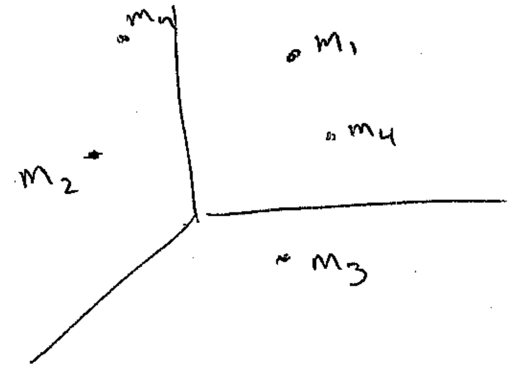
So,

$$\iiint_W dV = \int_0^{\frac{5}{2}} \int_0^{2\pi} \int_{3r^2-16}^{9-r^2} r \, dz\, d\theta\, dr = \frac{625\pi}{8}$$

$T(x,y,z)$ becomes $T(r,\theta,z) = zr^2$.

$$\iiint_W zr^2 \, r\,dz\,d\theta\,dr = \int_0^{\frac{5}{2}} \int_0^{2\pi} \int_{3r^2-16}^{9-r^2} zr^3 \, dz\,d\theta\,dr = \frac{-15625\pi}{256}$$

Center of mass:



center of mass:
depends on relative size of the m_i:
e.g. if m_1 is huge compared to rest the center of mass will be close to m_1

~~if m_1~~ weighted average of the coordinates of the locations of the masses.

~~Ex~~

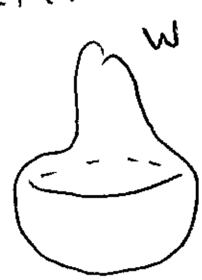
Defn: Given a system of n point masses $m_1, \dots, m_n$ at positions $(x_1, y_1, z_1), \dots, (x_n, y_n, z_n)$ the center of mass is at $(\bar{x}, \bar{y}, \bar{z})$ where

$$\bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i}, \quad \bar{y} = \frac{\sum_{i=1}^{n} m_i y_i}{\sum_{i=1}^{n} m_i}, \quad \bar{z} = \frac{\sum_{i=1}^{n} m_i z_i}{\sum_{i=1}^{n} m_i}.$$

Note: Check if $m_1 \ggg m_i$ then $(\bar{x}, \bar{y}, \bar{z}) \approx (x_1, y_1, z_1)$

Note: if the masses are in the plane, $\bar{z} = 0$, if they're on ~~the~~ the x-axis $\bar{y}, \bar{z} = 0$.

~~What~~ What if we have not points, but an object with mass density $\delta(x,y,z)$.



What is its center of mass?
The denominator will be the total mass of the object:

$$\iiint_W \delta(x,y,z)\, dV$$

The numerator of the x term should be "the integral over every point of mass at that point times the x coordinate". i.e.,

$$\iiint_W \underbrace{x}_{\text{coordinate}} \underbrace{\delta(x,y,z)\, dV}_{\text{mass}}$$

So,

$$\bar{x} = \frac{\iiint_W x\,\delta(x,y,z)\,dV}{\iiint_W \delta(x,y,z)\,dV}, \quad \bar{y} = \frac{\iiint_W y\,\delta(x,y,z)\,dV}{\iiint_W \cancel{y}\,\delta(x,y,z)\,dV}$$

$$\bar{z} = \frac{\iiint_W z\,\delta(x,y,z)\,dV}{\iiint_W \delta(x,y,z)\,dV}$$

Vocabulary:
- If $\delta(x,y,z)$ is constant, $(\bar{x},\bar{y},\bar{z})$ is called the centroid of W.
- ~~If W is~~ A two dim'l ~~region~~ region with varying density is called a lamina.
- The ~~values $\iiint_W x\,\delta(x$~~ numerators of $\bar{x},\bar{y},\bar{z}$ are called the first moments of W.

Moments of inertia info can be found in book.

Probability:

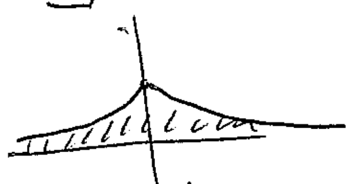
Def. A probability density function of a single variable is any function $f(x)$ s.t. $f(x) \geq 0$ for all $x \in \mathbb{R}$ and $\int_{-\infty}^{\infty} f(x) = 1$

e.g $\frac{1}{\sqrt{\pi}} e^{-x^2}$

Given such an f,

$$\text{Prob}(a \leq x \leq b) = \int_a^b f(x)\,dx.$$

e.g Check that $f(x) = e^{-2|x|}$ is a probability density fcn.



so $f(x) \geq 0$.

$$\int_{-\infty}^{\infty} e^{-2|x|} = \int_0^{\infty} e^{-2x}\,dx + \int_{-\infty}^{0} e^{-2(-x)}\,dx$$

$$= 2\int_0^{\infty} e^{-2x}\,dx = 2 \lim_{c\to\infty} \int_0^c e^{-2x}\,dx$$

$$= 2 \lim_{c\to\infty} \left(\underbrace{-\tfrac{1}{2}e^{-2c}}_{\to 0} - -\tfrac{1}{2}e^{0} \right)$$

$$= 2 \cdot \tfrac{1}{2} = 1.$$