Stokes's Theorem:

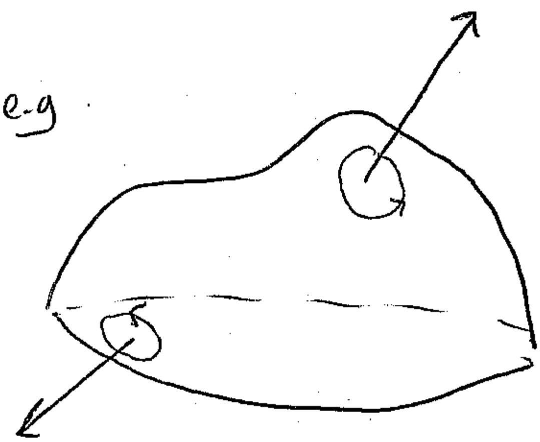
Vector line integrals ⟷ Surface integrals.

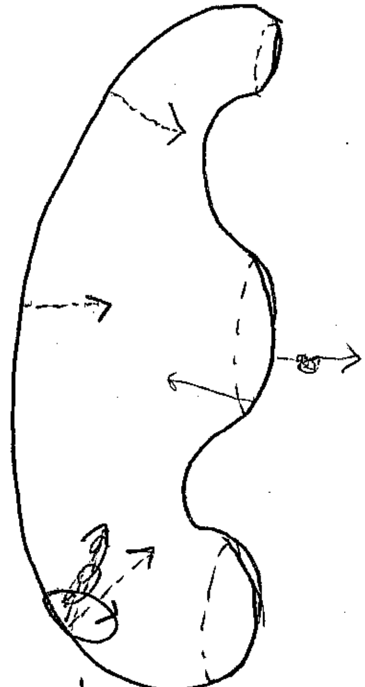
Both these kinds of integrals are orientation-dependent:

so we need to be careful about the orientations of the curve and surface:

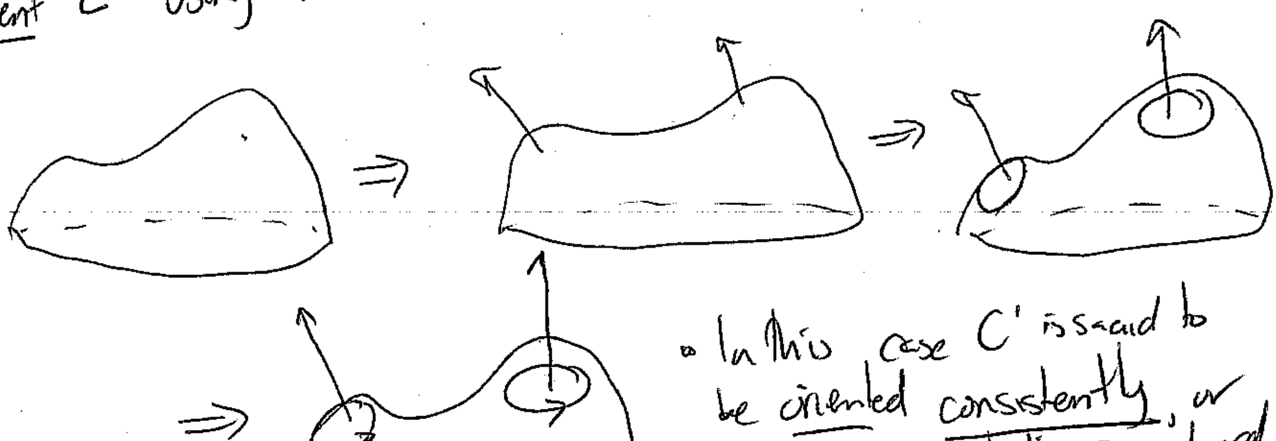
- Let S be a bounded, piecewise smooth, oriented surface in $\mathbb{R}^3$

e.g.





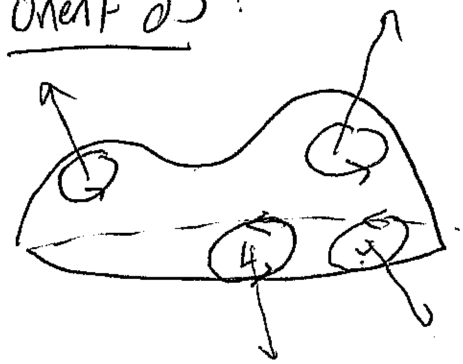
- Let C' be any simple, closed curve lying in S. Consider the unit ~~vector~~ normal vector $\hat{n}$ that indicates the orientation of S at any point inside C'.
- Orient C' using $\hat{n}$ and the right hand rule.



- In this case C' is said to be oriented consistently, or that C''s orientation is induced from that of S.

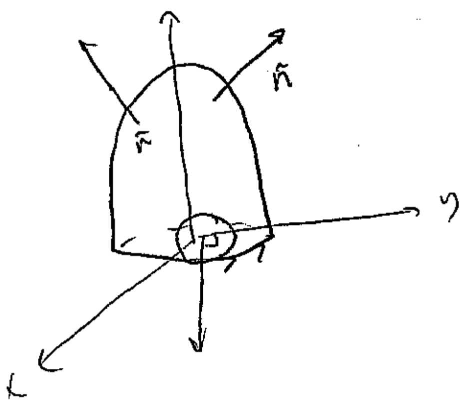
- ~~Orient ∂S in the same, also way.~~

Orient ∂S:

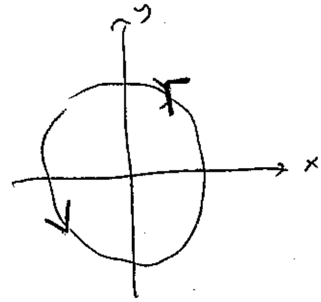


oriented curves that touch ∂S give an orientation ~~to~~ ∂S.

e.g ~~Sx~~ $z = 9 - x^2 - y^2$ paraboloid. S is the piece of the paraboloid above the xy-plane oriented with outward normal vector $\hat{n}$



1. Orient ∂S
2. Parameterize ∂S:
~~a circle of~~
from the x-axis to the y-axis
$\gamma = (3\cos t, 3\sin t, 0)$



Stokes's theorem: Let S be a bounded, piecewise ~~conts~~smooth, oriented surface in $\mathbb{R}^3$. Suppose that ∂S consists of finitely many piecewise smooth, simple closed curves oriented consistently with S. Let F be a vector field on $\mathbb{R}^3$ ~~on which S~~ defined on S. Then

$$\iint_S \nabla \times F \cdot dS = \oint_{\partial S} F \cdot ds$$

Verify Stokes's Theorem with S as above:

Let $F = (2z-y)\hat{i} + (x+z)\hat{j} + (3x-2y)\hat{k}$.

<u>Check</u>: $\nabla \times F = (-3,-1,2)$

$N = \nabla(z+x^2+y^2-9) = \{2x\hat{i} + 2y\hat{j} + \hat{k}$

$$\iint_S \nabla \times F\,dS = \iint_D (-3,-1,2)\cdot(2x,2y,1)\,dx\,dy$$

D is the circle of radius 3 in the xy-plane

$$= \iint_D -6x-2y+2\,dx\,dy$$

$$= \iint_D -6x\,dx\,dy + \iint_D -2y\,dx\,dy + \iint_D 2\,dx\,dy$$

$$= 0 + 0 + 2\,\text{Area}(D)$$

since $-6x$ is odd and D is symm.

$$= 2\pi 3^2 = 18\pi.$$

OTOH

$$\oint_{\partial S} F\cdot ds = \int_0^{2\pi} F(\sigma(t))\cdot\sigma'(t)\,dt$$

$$= \int_0^{2\pi} (-3\sin t, 3\cos t, 9\cos t - 6\sin t)\cdot(-3\sin t, 3\cos t, 0)\,dt$$

$$= \int_0^{2\pi} 9\sin^2 t + 9\cos^2 t\,dt = 9\int_0^{2\pi} dt = 18\pi.$$

E.g Pretty awesome example:

with outward normal vector

S: $z = e^{-(x^2+y^2)}$ for $z \geq \frac{1}{e}$ defined over the unit disk. D.

F: $(e^{y+z} - 2y)\hat{i} + (x e^{y+z} + y)\hat{j} + e^{x+y}\hat{k}$

$\nabla \times F = (e^{x+y} - x e^{y+z})\hat{i} + (e^{y+z} - e^{x+y})\hat{j} + 2\hat{k}$

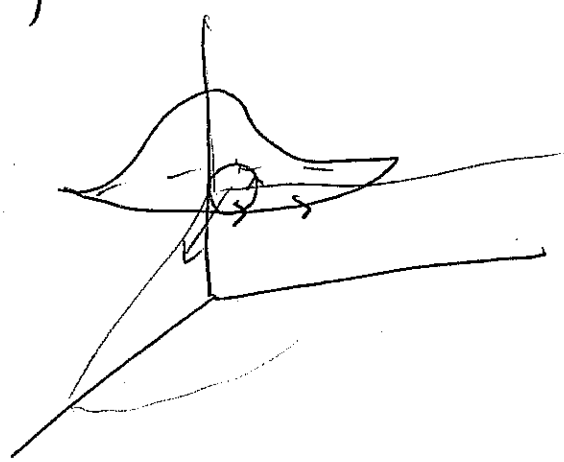
$N = \nabla f = 2x e^{-(x^2+y^2)}\hat{i} + 2y e^{-(x^2+y^2)}\hat{j} + \hat{k}$

$$\iint_S \nabla \times F \cdot dS = \iint_D 2x(e^{-x^2+y^2})(e^{x+y} - x e^{y+z}) + 2y(e^{-x^2+y^2})(e^{y+z} - e^{x+y}) + 2 \, dx\,dy$$

Hard integral to do

Use Stokes's Thm in the obvious way

$$\iint_S \nabla \times F \cdot dS = \oint_{\partial S} F \cdot ds$$



1. orient ∂S:
2. Parameterize ∂S

$$\sigma(t) = \begin{cases} x = \cos t \\ y = \sin t \\ z = \frac{1}{e} \end{cases} \quad 0 \leq t \leq 2\pi$$

3. Compute $\oint_{\partial S} F \cdot ds$

$$\oint_{\partial S} F \cdot ds = \int_0^{2\pi} F(\sigma(t)) \cdot \sigma'(t)\, dt$$

$$= \int_0^{2\pi} 2\sin^2 t - \sin t\, e^{\sin t + 1/e} + \cos^2 t\, e^{\sin t + 1/e} + \cos t \sin t \, dt.$$

Hard integral to do

Use Stokes's Theorem to cool way.

Suppose there was some other surface S' so that $\partial S = \partial S'$ (in particular, they have the same orientation).

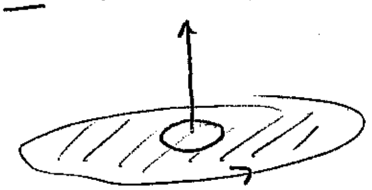
Then

$$\iint_S \nabla\times F \cdot dS = \oint_{\partial S} F\cdot ds = \oint_{\partial S'} F\cdot ds = \iint_{S'} \nabla\times F \cdot dS$$

So, if we can find another surface S' with the same boundary as S then $\iint_{S'} \nabla\times F\cdot dS = \iint_S \nabla\times F \cdot dS$

Can you think of an easy candidate?

A: The unit disk ~~at~~ ~~at~~ $\frac{1}{e}$ above the xy-plane with upward pointing normal vector.



S' $N = \hat{k}$.

Now: $\iint_S \nabla\times F \cdot dS = \iint_{S'} \nabla\times F\cdot dS$

$$= \iint_D \left(e^{x+y} - x\,e^{y+z}\right)\hat{\imath} + \left(e^{y+z} - e^{x+y}\right)\hat{\jmath} + 2\hat{k}\Big) \cdot (\hat{k})$$

$$= \iint_D 2\,dx\,dy = 2\,\text{Area}(\text{unit disk})$$

$$= 2\pi.$$