Check

$$T_s \times T_t = -a^2 \sin t \left(\cos s \sin t \, \hat{i} + \sin s \sin t \, \hat{j} + \cos t \, \hat{k} \right)$$

$$\| T_s \times T_t \| = a^2 \sin t$$

Surface area.

$$\int_0^{\pi} \int_0^{2\pi} -a^2 \sin t \, ds \, dt = -2\pi \int_0^{\pi} a^2 \sin t \, dt$$

$$= -2\pi \, a^2 \cos t \Big|_0^{\pi}$$

$$= -2\pi a^2 (-1 - 1)$$

$$= 4\pi a^2 .$$

Special case: Suppose z is a fcn of x and y, $z = f(x,y)$. Then The parameterization of Σ is $(x, y, f(x,y))$

Then
$$T_x = \frac{\partial x}{\partial x} \hat{i} + \frac{\partial f}{\partial x} \hat{k} = \hat{i} + \frac{\partial f}{\partial x} \hat{k}$$

$$T_y = \frac{\partial y}{\partial y} \hat{j} + \frac{\partial f}{\partial y} \hat{k} = \hat{j} + \frac{\partial f}{\partial y} \hat{k} .$$

Check
$$\| T_x \times T_y \| = \left\| \det \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & \frac{\partial f}{\partial x} \\ 0 & 1 & \frac{\partial f}{\partial y} \end{pmatrix} \right\|$$

$$= \left\| -\frac{\partial f}{\partial x} \hat{i} - \frac{\partial f}{\partial y} \hat{j} + \hat{k} \right\|$$

$$= \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}$$

So area $= \iint_D \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} \, dA$. Compare to arclength

Line integrals and surface integrals:

- $\int_\sigma F\,ds$ scalar line integral. $ds = \|\sigma'(t)\|\,dt$. ~~speed × time~~ $\frac{m}{s} \times s$

$\iint_\Sigma F\,dS$ - scalar surface integral $dS = \|T_s \times T_t\|\,ds\,dt$.
~~area~~/time $\frac{m^2}{s^2}\cdot s^2$

- $\int_\sigma ds$ = arclength — $\int_\sigma F\,ds = \int F(\sigma(t))\,\|\sigma'(t)\|\,dt$

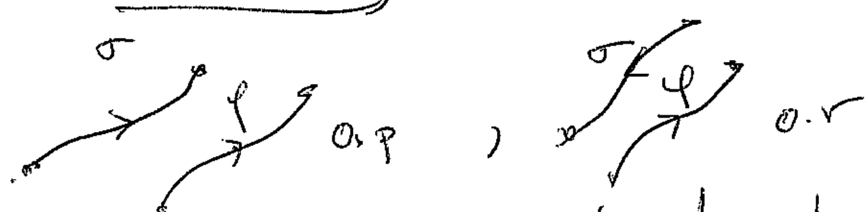
$\iint_\Sigma dS$ = surface area. $\iint_\Sigma F\,dS = \iint F(\Sigma(s,t))\,\|N(s,t)\|\,ds\,dt$.

- reparameterization

~~reparameterization of σ if there exi~~

Paths: ~~φ is a reparameterization~~

- $\varphi: [c,d] \to \mathbb{R}^n$ is a reparameterization of $\sigma: [a,b] \to \mathbb{R}^n$ if there exists some (1-1, onto) $u: [a,b] \to [c,d]$ s.t. $\varphi(u(t)) = \sigma(t)$
- φ is orientation preserving if $\varphi(c) = \sigma(a)$, $\varphi(d) = \sigma(b)$,
  " reversing if $\varphi(c) = \sigma(b)$ $\varphi(d) = \sigma(a)$.

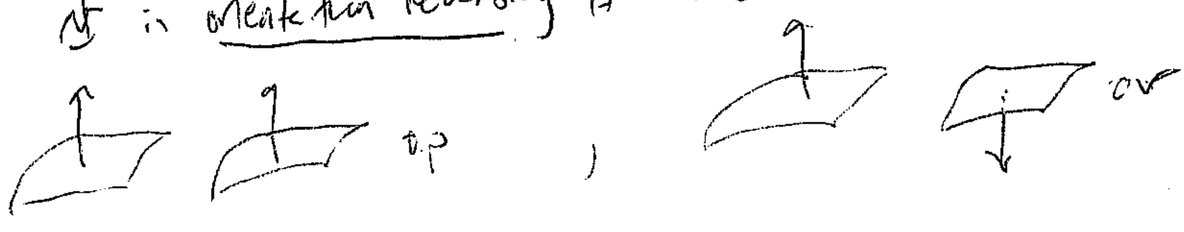


Surfaces:

$\Phi: D' \to \mathbb{R}^n$ is a reparameterization of $\Sigma: D \to \mathbb{R}^n$ if there exists some (1-1 & onto) $U: D \to D'$ s.t. $\Phi \circ U(s,t) = \Sigma(s,t)$.

Φ is orientation-preserving if the Jacobian of U is positive

Φ is orientation reversing if the Jacobian of U is negative



- Reparameterizations of surfaces and paths have no effect on ~~the~~ surface/line integrals.

e.g. The helicoid parameterized by

$$\Sigma(s,t) = (s\cos t, s\sin t, t), \quad 0 \le s \le 1, \ 0 \le t \le \frac{\pi}{2}$$

<u>Note</u>: ~~Also~~, $\iint_\Sigma f\, dS = \cancel{\int_0^{1}\int_0^{2\pi} t^3 \sqrt{\ }}$

$$= \iint_D f(\Sigma(s,t))\, \|N(s,t)\|\, ds\, dt$$

$T_s = \cos t\, \hat{\imath} + \sin t\, \hat{\jmath} + 0\hat{k}$

$T_t = -s\sin t\, \hat{\imath} + s\cos t\, \hat{\jmath} + \hat{k}$

$$N(s,t) = T_s \times T_t = \det\begin{pmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \cos t & \sin t & 0 \\ -s\sin t & s\cos t & 1 \end{pmatrix} = \hat{\imath}(\sin t) - \hat{\jmath}(\cos t) + \hat{k}\,(s\cos^2 t + s\sin^2 t)$$

$$\|N(s,t)\| = \sqrt{\sin^2 t + \cos^2 t + (s(\cos^2 t + \sin^2 t))^2} = \sqrt{1+s^2}$$

If $f = x$, then

$$\iint_\Sigma f\, dS = \int_0^{\frac{\pi}{2}}\int_0^{1} f(\Sigma)\sqrt{1+s^2}\, ds\, dt$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{1} (s\cos t)\sqrt{1+s^2}\, ds\, dt$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{3}(1+s^2)^{3/2}\Big|_0^1 \cos t\, dt$$

$$= (2^{3/2}-1)\int_0^{\frac{\pi}{2}} \cos t\, dt = (2^{3/2}-1)\sin t\Big|_0^{\frac{\pi}{2}} = (2^{3/2}-1).$$

Note Σ can be reparameterized as

$$\tilde{\Sigma}(s,t) = \left(\frac{s}{2}\cos 2t, \frac{s}{2}\sin 2t, 2t\right) \text{ for } 0 \le t \le \frac{\pi}{4}, \ 0 \le s \le 2$$

Is Φ orientation reversing or preserving?

OK ~~$D' = [0, \frac{\pi}{4}] \times [0, \frac{1}{2}]$ $D = [0, \frac{\pi}{2}] \times [0, 1]$~~

$\mathcal{U}: D \to D'$ given by

$\mathcal{U}(s,t) = \left(\frac{s}{2}, 2t\right)$ Jacobian of $\mathcal{U}$ $\cdot \det\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{pmatrix} = 1$

So it's orientation ~~rever~~ preserving.

---

vector line integrals

- $\int_\sigma F \cdot ds = \int F(\sigma(t)) \cdot \sigma'(t)\, dt$

$\iint_{\Sigma} F \cdot dS = \iint F(\Sigma(t)) \cdot N(s,t)\, ds\, dt$

- $\int_\sigma F \cdot ds = \int_\sigma \overbrace{(F \cdot T)}^{\text{scalar integral}}\, ds$ measures tangential flow along σ

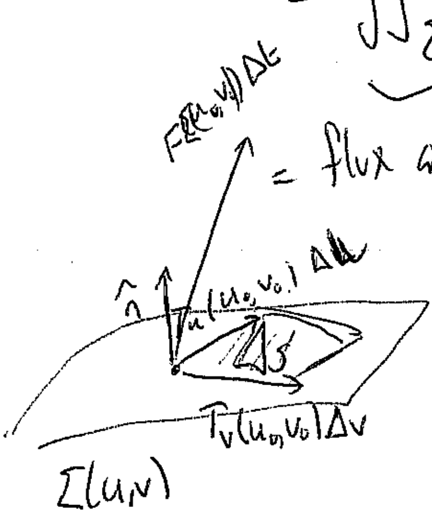
$$\iint_\Sigma F \cdot dS = \iint_D F(\Sigma(s,t)) \cdot N(s,t)\, ds\, dt$$

$$= \iint_D F(\Sigma(s,t)) \cdot \hat{n}(s,t)\, \|N(s,t)\|\, ds\, dt$$

where $\hat{n}$ is the unit normal vector to Σ.

$$= \underbrace{\iint_\Sigma (F \cdot \hat{n})\, ds\, dt}_{\text{scalar line integral}}$$

$=$ flux across Σ.



Amount of fluid moving across Σ $\approx$ volume of the parallelepiped

$=$ (height)(area of base)

$= F(\Sigma(u_0,v_0))\Delta t \cdot \hat{n}(u_0,v_0)$

$$\text{Flux} = \frac{\text{fluid across } \Sigma}{\text{time}} = \iint_\Sigma (F \cdot \hat{n})\, dS$$

- If φ is an orientation reversing reparameterization of σ then

$$\int_\sigma F \cdot ds = -\int_\varphi F \cdot ds$$

If φ is an orientation preserving reparameterization of σ, then

$$\int_\sigma F \cdot ds = \int_\varphi F \cdot ds.$$

- If Φ is an orientation reversing reparameterization of Σ, then

$$\iint_\Phi F \cdot dS = -\iint_\Sigma F \cdot dS$$

- If Φ is an orientation preserving reparameterization of Σ, then

$$\iint_\Phi F \cdot dS = \iint_\Sigma F \cdot dS.$$

e.g. Let S be the sphere $x^2+y^2+z^2=a^2$, $F(x,y,z) = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$.
Orient S by outward pointing unit normal vectors.
Parameterize S by $\Sigma(s,t) = (a \cos s \sin t, a \sin s \sin t, a \cos t)$
$0 \le s \le 2\pi$, $0 \le t \le \pi$.

Check: $N(s,t) = -a^2 \sin t\,(\cos s \sin t\,\hat{\imath} + \sin s \sin t\,\hat{\jmath} + \cos t\,\hat{k})$

$\hat{n}(s,t) = \dfrac{N(s,t)}{\|N(s,t)\|} = -(\cos s \sin t\,\hat{\imath} + \sin s \sin t\,\hat{\jmath} + \cos t\,\hat{k})$.
inward pointing unit vectors.

Σ has opposite orientation of S

$$\iint_S F \cdot dS = -\iint_\Sigma F \cdot dS = -\int_0^\pi \int_0^{2\pi} F(\Sigma(s,t)) \cdot N(s,t)\,ds\,dt$$

$$= -\int_0^\pi \int_0^{2\pi} (a \cos s \sin t, a \sin s \sin t, a \cos t) \cdot (-a^2 \sin t\,(\cos s \sin t, \sin s \sin t, \cos t))\,ds\,dt$$

$$= a^3 \int_0^{\pi}\int_0^{2\pi} \sin t \, ds\, dt = 4\pi a^3$$

But we can also do it geometrically: (This often works if S is a level curve of a graph of a function $f(x,y)$.)

~~Note~~ $F(x,y,z) = x\hat{i} + y\hat{j} + z\hat{k}$

<u>Note:</u> $x^2+y^2+z^2 = a^2$, $\hat{n} = \dfrac{x\hat{i}+y\hat{j}+z\hat{k}}{a}$ outward pointing unit vector.

Now $$\iint_S (F \cdot \hat{n})\, dS = \iint x\hat{i}+y\hat{j}+z\hat{k} \cdot \left(\frac{x\hat{i}+y\hat{j}+z\hat{k}}{a}\right) dS$$

$$= \iint \frac{x^2+y^2+z^2}{a}\, dS$$

$$= a \iint dS = a(\text{surface area of } S)$$
$$= 4\pi a^3.$$

<u>Warning:</u>

~~Line~~ in Every path has an orientation

Not true for surfaces. E.g. Möbius strip

One sided ~~and~~ so ~~they~~ same points has two distinct unit vectors:

Parameterization

$$\Sigma(s,t) = \left(\left(1+t\cos\frac{s}{2}\right)\cos s,\ \left(1+t\cos\frac{s}{2}\right)\sin s,\ t\sin\frac{s}{2}\right)$$

$$-\frac{1}{2} \le t \le \frac{1}{2} \qquad 0 \le s \le 2\pi$$

But $N(0,0) = N(2\pi,0)$

even though $\Sigma(2\pi,0) = \Sigma(0,0)$.