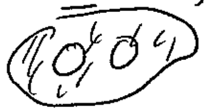
Def.: A region D is <u>connected</u> if any point can be reached from any other along some curve in D.

connected not connected

A region D is <u>simply connected</u> if every path between any two points contains points only in D

Simply connected

not simply connected.



Thm: Let $F = M(x,y)\,\hat{\imath} + N(x,y)\,\hat{\jmath}$ be a vector field on a simply connected region D. Suppose M and N are smooth in D so that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \text{for all } (x,y)\in D.$$

Then F is <u>conservative</u>

~~e.g. $F(x,y) = (x-y)\hat{\imath} + (x-2)\hat{\jmath}$ is conservative? why?~~

~~If F conservative, then $\frac{\partial M}{\partial y} = \frac{\partial M}{\partial x}$~~

Thm: ~~If $F = M(x,y)\,\hat{\imath} + N(x,y)\,\hat{\jmath}$~~ is s.t. $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ on D, then F is ~~conservative~~. i.e., if $\oint_{\partial D}$ ~~$F\cdot ds = \iint_D$~~ $= 0$ then F is conservative

e.g. $F(x,y) = (x-y)\,\hat{\imath} + (x-2)\,\hat{\jmath}$ is conservative

$F(x,y) = (3+2xy)\,\hat{\imath} + (x^2-3y^2)\,\hat{\jmath}$ is not conservative.

Corollary: "$\oint F\cdot ds = 0 \iff F$ conservative"

e.g. Let $F = (2xy + \cos 2y)\,\hat{\imath} + (x^2 - 2x\sin 2y)\,\hat{\jmath}$.
Let C be the ellipse $x^2 + \frac{y^2}{4} = 1$. Check F is conservative

$\rightarrow \oint_C F\cdot ds = 0$. w/out integrating

Vector fields in $\mathbb{R}^3$:

Old Thm: For D, $F = \nabla f \Rightarrow \text{curl } F = 0$

New Thm: ~~let~~ F ~~be~~ defined on all of $\mathbb{R}^3$. If curl $F = 0$, then F is conservative.

e.g. $F(x,y,z) = xz\hat{i} + xyz\hat{j} - y^2\hat{k}$ is not conservative

$F(x,y,z) = y^2z^3\hat{i} + 2xyz^3\hat{j} + 3xy^2z^2\hat{k}$ is conservative.

Skip

Status:

we ~~know~~ can tell ~~if~~ when $F: \mathbb{R}^2 \to \mathbb{R}^2$ and $F: \mathbb{R}^3 \to \mathbb{R}$ is such that

$F = \nabla f$ for some f. Can we find the f?

we've done this before, but let's review:

e.g. $F(x,y,z) = y^2z^3\hat{i} + 2xyz^3\hat{j} + 3xy^2z^2\hat{k}$

Let $F = \nabla f \left(\frac{\partial}{\partial x} f, \frac{\partial}{\partial y} f, \frac{\partial}{\partial z} f \right)$

Then $\frac{\partial}{\partial x} f = y^2z^3$ $\quad \frac{\partial}{\partial y} f = 2xyz^3$ $\quad \frac{\partial}{\partial z} f = 3xy^2z^2$

$f(x,y,z) = xy^2z^3 + g(y,z)$; $\frac{\partial}{\partial y} f(x,y,z) = 2xyz^3 + \frac{\partial}{\partial y} g(y,z)$

compare with . Then $\frac{\partial}{\partial y} g(y,z) = 0$, so ~~g~~ then $g(y,z) = h(z)$.

$f(x,y,z) = xy^2z^3 + h(z)$

$\frac{\partial f}{\partial z} = 3xy^2z^2 + h'(z)$. Thus $h'(z) = 0$. So let

$f(x,y,z) = xy^2z^3 + C$ for some constant C.

Parameterized surfaces:

To visualize ~~the~~ surfaces in 3 space...

1. $z = f(x,y)$ — function
2. $f(x,y,z) = k$ — level set

Now we introduce a third way, we can parameterize a surface:

<u>Def'n</u>: Let D be a region in $\mathbb{R}^2$. A parameterized surface in $\mathbb{R}^3$ is a continuous function $\Sigma : D \to \mathbb{R}^3$ that is 1-1 on D.

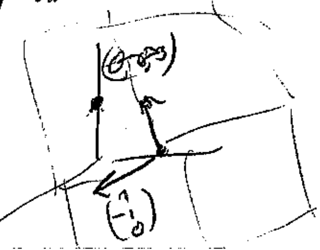
We refer to the image $\Sigma(D)$ as the <u>underlying surface of</u> Σ.

i.e.,

Like with paths, $\Sigma = (x(s,t), y(s,t), z(s,t))$ for all $(s,t) \in D$.

<u>eg</u>
$$\Sigma(s,t) = s(\hat{i} - \hat{j}) + t(\hat{i} + 2\hat{k}) + 3\hat{j}$$
$$= (s+t, 3-s, 2t)$$

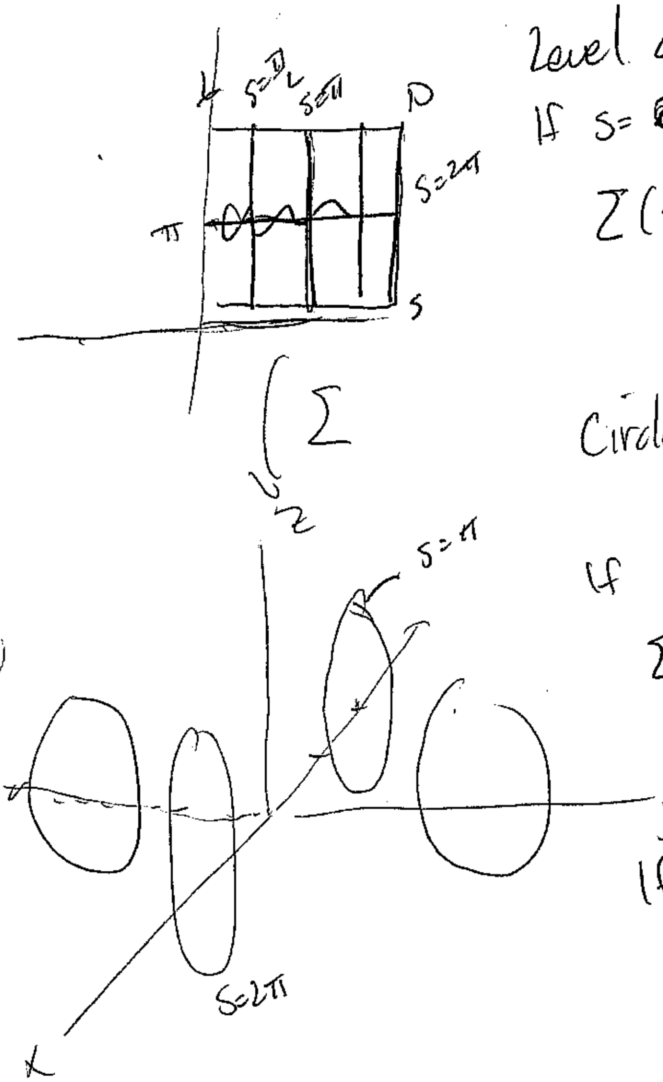
Notice that $\Sigma(s,t)$ is the ~~image~~ plane spanned by $\begin{pmatrix}1\\-1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\2\end{pmatrix}$ and the vector $\begin{pmatrix}0\\3\\0\end{pmatrix}$



eg Graph

$$\Sigma(s,t) = \left((2+\cos t)\cos s,\ (2+\cancel{\sin t}\ \cos t)\sin s,\ \sin t \right)$$

$0 \le s, t \le 2\pi$



Level curve analogue:

If $s = \pi$

$$\Sigma(s,t) = \left((2+\cos t)(\cos\pi),\ (2+\sin t)\sin\pi, \sin t\right)$$

$$= \left(-(2+\cos t),\ 0,\ \sin t\right)$$

Circle of radius 1, centered at $(-2,0,0)$, $\parallel$ the xz plane.

If $s = 2\pi$

$$\Sigma(s,t) = \left((2+\cos t), 0, \sin t\right)$$

Circle of radius 1 centered at $(2,0,0)$ $\parallel$ xz plane

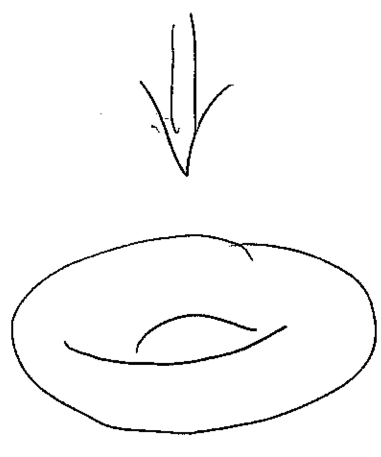
If $s = \frac{\pi}{2}$

$$\left((2+\cos t)(0),\ (2+\cos t),\ \sin t\right)$$

circle of radius 1 at $(0,2,0)$ parall yz-plane

If $s = \frac{3\pi}{2}$ $\Rightarrow$ circle of radius 1 at $(0,-2,0)$

For any s you're going to some circle.



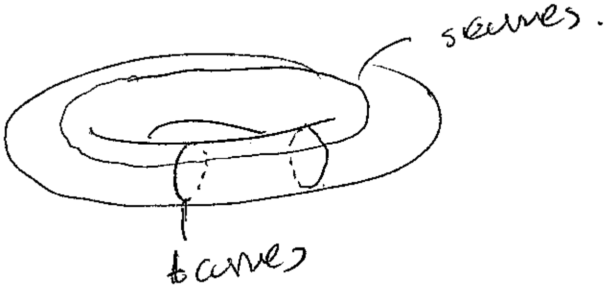
a torus.

Def'n Fixing s gives a t coordinate curve

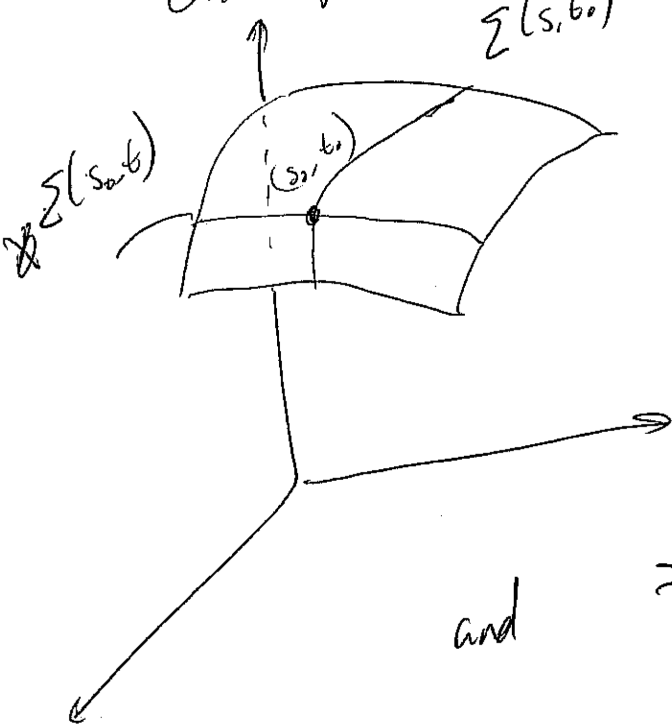
$\Sigma(s_0,t)$, a function of t

Fixing t gives an s coordinate curve

$\Sigma(s,t_0)$, a function of s.



Once you have curves, we can talk about tangent vectors:



At the point (s_0,t_0)

define

$$\vec{T}_s(s_0,t_0) = \frac{\partial \Sigma}{\partial s}(s_0,t_0)$$

$$= \frac{\partial x}{\partial s}(s_0,t_0)\hat{i} + \frac{\partial y}{\partial s}(s_0,t_0)\hat{j} + \frac{\partial z}{\partial s}(s_0,t_0)\hat{k}$$

and $$\vec{T}_t(s_0,t_0) = \frac{\partial \Sigma}{\partial t}(s_0,t_0)$$

$$= \frac{\partial x}{\partial t}(s_0,t_0)\hat{i} + \frac{\partial y}{\partial t}(s_0,t_0)\hat{j} + \frac{\partial z}{\partial t}(s_0,t_0)\hat{k}$$

Let

$$N(s_0,t_0) = T_s(s_0,t_0) \times T_t(s_0,t_0)$$

be the normal vector to the surface at (s_0,t_0).

Def: $S=\Sigma(D)$ is smooth at $\Sigma(s_0,t_0)$

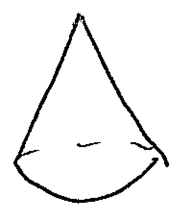
if $N(s_0,t_0) \neq 0$.

Intuition: smooth should make you think of "no sharp edges"

e.g.

not smooth along its edges

smooth everywhere

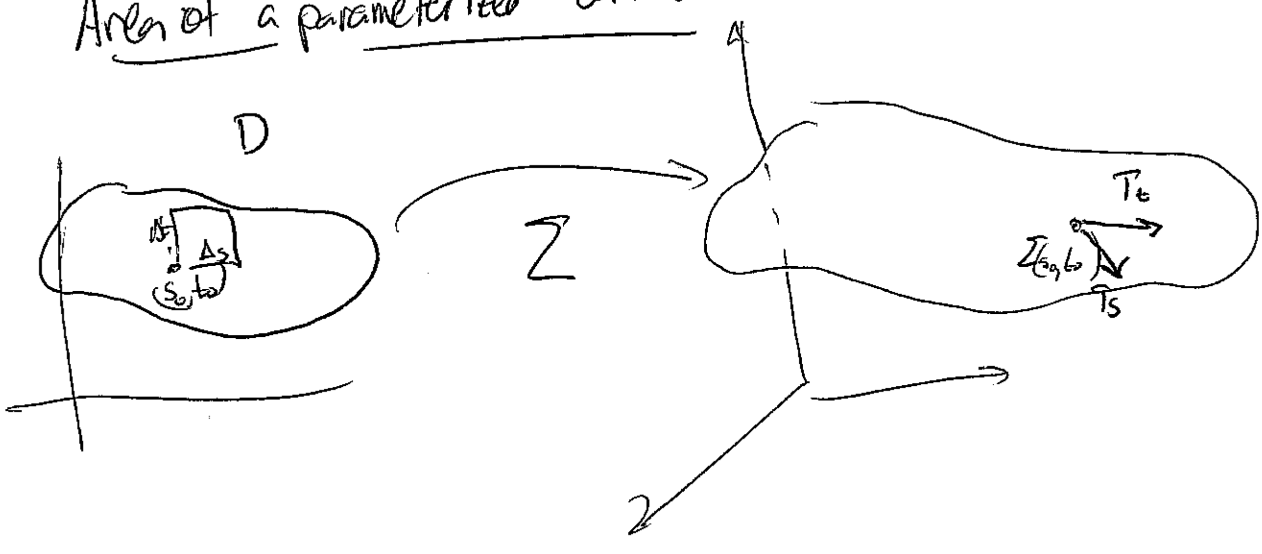


not smooth at its point.

Note at non smooth parts, no well-defined normal vector

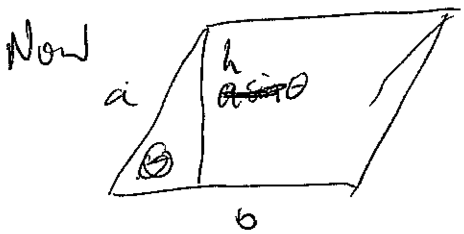
(think of $y=|x|$.)

Area of a parameterized surface:



since for a little time interval the s- and t- curves are following their tangent vectors

Now



$\sin\theta = \frac{h}{a} \Rightarrow a\sin\theta = h$

Area of parallelogram

$\approx ab\sin\theta = \|\vec{T}_t \times \vec{T}_s\|.$

So, surface area of $S =$

$$\iint_D \|T_t \times T_s\| \, ds\, dt.$$

e.g. Prove that the surface area of a sphere of radius a is $4\pi a^2$.

Note a sphere of radius a has a parameterization

$$\Sigma(s,t) = (a\cos s \sin t,\ a \sin s \sin t,\ a\cos t)$$

(think $t = \varphi$, $s = \theta$)

for $s,t \in [0,2\pi]\times[0,\pi]$

$$T_t(s,t) = \frac{\partial \Sigma}{\partial t}(s,t) = \frac{\partial\, a\cos s\sin t}{\partial t}\hat{\imath} + \frac{\partial\, a\sin s\sin t}{\partial t}\hat{\jmath} + \frac{\partial\, a\cos t}{\partial t}\hat{k}$$

$$= \cancel{a\cos^2 t}\ a\cos s\cos t\,\hat{\imath} + a\sin s\cos t - a\sin t$$

$$T_s(s,t) = -a\sin s\sin t\,\hat{\imath} + a\cos s\sin t\,\hat{\jmath} + 0\hat{k}$$