Other ways of thinking about Green's Theorem.

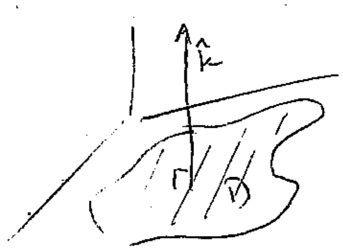
If $F = M(x,y)\hat{i} + N(x,y)\hat{j}$

$$\nabla \times F = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\hat{k}.$$

Since $\hat{k}\cdot\hat{k} = 1$,

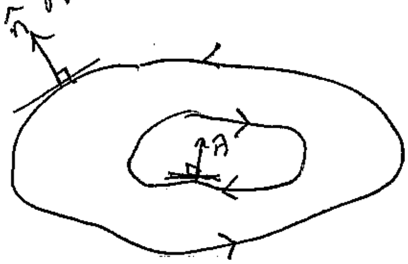
<u>Thm</u>: $$\oint_{\partial D} F\cdot ds = \iint_D (\nabla\times F)\cdot\hat{k}\, dA$$

- Interpretation: If D is in the plane, then $\hat{k}$ is the unit normal vector to D.
  Then $\nabla\times F\cdot\hat{k}$ is the component of the curl normal to D.
  This will generalized = Stokes's Theorem.



- <u>Greens Thm and Divergence</u>

Now suppose $\hat{n}$ is the unit normal vector pointing away from D

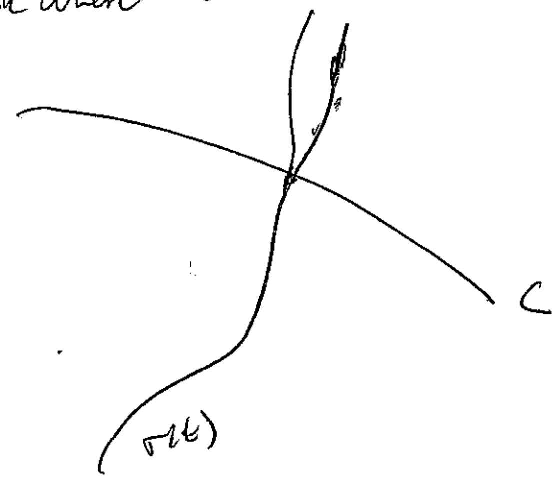


<u>Thm</u>: D and $\hat{n}$ as above.

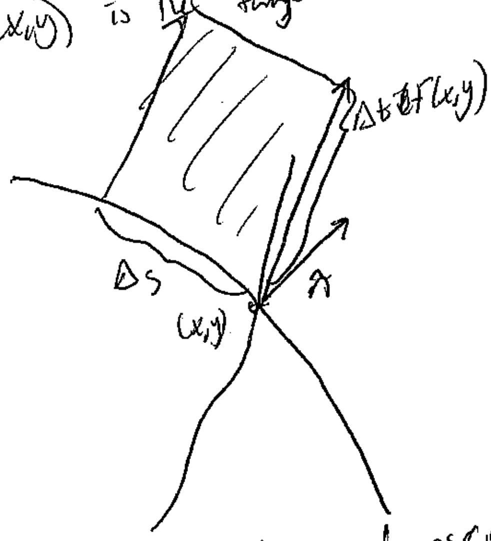
$\cancel{\oint_{\partial D} F\cdot ds = \oint \iint_D}$ $\quad \underbrace{\oint F\cdot\hat{n}\,ds}_{\text{scalar line integral}} = \iint_D \nabla\cdot F\, dA$

Interpretation:

- vector field has flow lines $\sigma(t)$.
- at (x,y) $\sigma(t)$ has tangent vector $F(x,y)$
- look where $\sigma(t)$ crosses the curve C



- time Δt after $\sigma(t)$ is at (x,y), the particle is at $F(x,y)\Delta t$ since $F(x,y)$ is the tangent vector.



- The ~~flux~~ total fluid moved across C is the area of the parallelogram

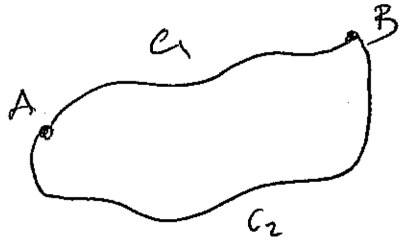
$$\approx \underbrace{(\Delta s)}_{\text{base}}\underbrace{\left(F(x,y)\Delta t \cdot \hat{n}\right)}_{\text{height.}}$$

- The total fluid moved across would be an integral.
- The flux ~~of~~ is the total fluid moved across divided by the elapsed time

$$\int_C F \cdot n \, ds$$

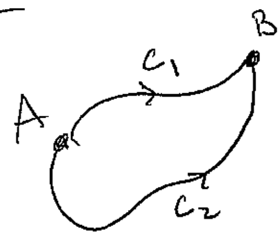
Independence of Path:

$\int_C F \cdot ds$ is independent of path if $\int_{C_1} F \cdot ds = \int_{C_2} F \cdot ds$ for any two curves between A and B.



Thm: $\int_C F \cdot ds$ is independent of path in D if and only if $\oint_C F \cdot ds = 0$ for every closed path in D. iff $\iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = 0$

Idea of Proof: ~~Suppose~~ $\int_C F \cdot ds$ is independent of path. Then



$\int_{C_1} F \cdot ds = \int_{C_2} F \cdot ds$

$\int_{C_1} F \cdot ds - \int_{C_2} F \cdot ds = 0$

Change orientation of C_2 to get a closed loop $0 = \int_{C_1} F \cdot ds + \int_{-C_2} F \cdot ds$

$= \oint_C F \cdot ds.$

Conservative vector field:
Same as a gradient vector field, ie,
$F = \nabla f$ for some f.

Vector fields on $\mathbb{R}^2$

Old Thm: If F a gradient (conservative) vector field then $\operatorname{curl} F = 0$

New Thm: If $\int_C F \cdot ds$ is independent of path in D, then F is a conservative vector field

So We can show easily, when F is not a conservative vector field.
We want a better way to show when F is a conservative vector field.