Greens Theorem:

vector line integral around a closed curve in $\mathbb{R}^2$ $\Longleftrightarrow$ a double integral over the region D bounded by the curve.

(Greens Theorem)

Thm: D ~~close~~ $\subseteq \mathbb{R}$ closed and bounded, $\partial D = C$ (boundary of D) consists of finitely many simple, closed curves.

orient C so that D is on the left as ~~you~~ one traverses C

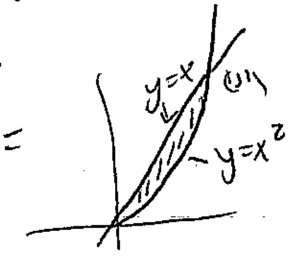
$F(x,y) = M(x,y)\,\hat{i} + N(x,y)\,\hat{j}$. defined on D. Then

$$\left(\oint_{\partial D = C} F \cdot ds = \right) \oint M\,dx + N\,dy$$
$$= \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy.$$

Note: ~~N and M are~~ 2 sides differ by a derivative and dimension

e.g Verify Green's Theorem:

$F = xy\,\hat{i} + y^2\,\hat{j}$, D =



$C = \partial D$: need to orient ∂D as

since D needs to be on the left

$C = C_1 + C_2$ where

$C_1 = (t, t^2),\ 0 \le t \le 1 \qquad C_2 = (1-t, 1-t),\ 0 \le t \le 1.$

$$\oint_C xy\,dx + y^2 dy = \int_{C_1} xy\,dx + y^2 dy + \int_{C_2} xy\,dx + y^2 dy$$

$$= \int_0^1 \left(t t^2 + (t^2)^2\, 2t\right) dt + \int_0^1 \left((1-t)(1-t) + (1-t)^2\right)(-dt)$$

$$= \frac{-1}{12}.$$

OTOH

$$\iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dy\,dx = \iint_D \left(\frac{\partial}{\partial x} y^2 - \frac{\partial}{\partial y} xy\right) dy\,dx$$

$$= \int_0^1 \int_{x^2}^{x} -x\,dy\,dx$$

$$= -\frac{1}{12}.$$

<u>Thm</u>: Let D be as above. Then

$$\boxed{\text{Area of } D = \frac{1}{2} \oint_{\partial D} -y\,dx + x\,dy.}$$

<u>Pf</u>: Area of $D = \iint_D dx\,dy = \frac{1}{2} \iint_D 2\,dx\,dy$

$$= \frac{1}{2} \iint_D 1 - (-1)\,dx\,dy$$

$$= \frac{1}{2} \iint_D \frac{\partial x}{\partial x} - \frac{\partial(-y)}{\partial y}$$

By Green's Theorem

$$= \frac{1}{2} \oint_{\partial D} -y\,dx + x\,dy.$$

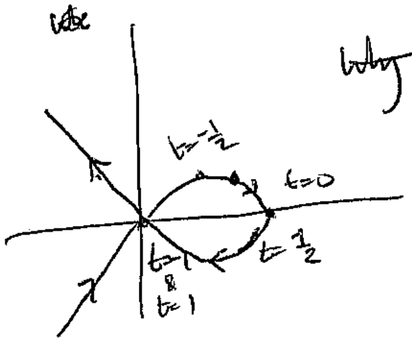
eg (a) Sketch the parameterized curve

$\sigma(t) = (1-t^2, t^3-t)$ with its orientation.

When are the x-coordinates positive:

when $1-t^2 \geq 0$ $\quad 1 \geq t^2$, $\quad 1 \geq t \geq -1$

~~when are y-coordinates positive~~



Why ~~False? We can use~~

(b) Find the area enclosed by the closed curve:

Why is this false.. Because σ has the wrong orientation.

$$\text{area} = \frac{1}{2}\oint_{\partial D \text{ as drawn}} -y\,dx + x\,dy \text{ ?}$$

we need to reverse its orientation: Recall that

if $\sigma : [a,b] \to \mathbb{R}^n$, then its opposite is $\sigma_{opp}(t) = \sigma(a+b-t)$.

In this case $a=-1$, $b=1$ so $\sigma_{opp}(t) = \sigma(-t)$.

Then in our case $\sigma_{opp}(t) = (1-t^2, -t^3+t)$

$$\text{area} = \frac{1}{2}\oint_{\partial D \text{ w/ reverse orientation}} -(-t^3+t)(-2t\,dt) + (1-t^2)(-3t^2+1)\,dt$$

$$= \frac{8}{15}.$$