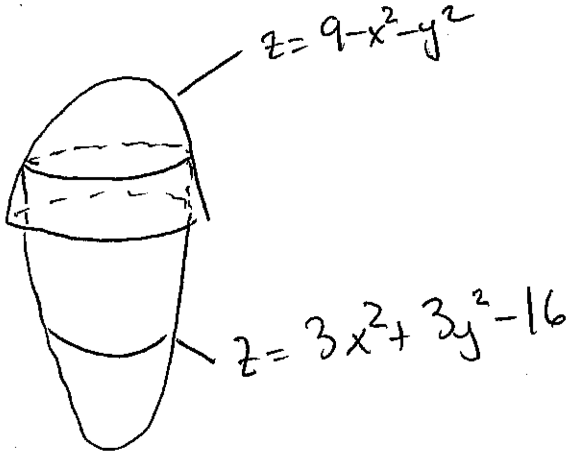
Set up the volume integral for



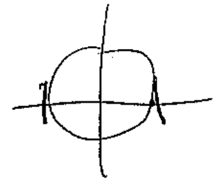
Notice

$3x^2 + 3y^2 - 16 \le z \le 9 - x^2 - y^2$

The figure is widest at the intersection

$3x^2 + 3y^2 - 16 = 9 - x^2 - y^2$

$4x^2 + 4y^2 = 25$. So



$\sqrt{\frac{25}{4} - x^2} \le y \le \sqrt{\frac{25}{4} - x^2}$

and $-\frac{5}{2} \le x \le \frac{5}{2}$.

So

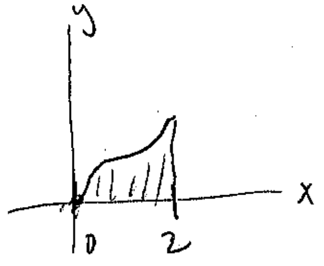
$$\iiint_W 1\, dV = \int_{-\frac{5}{2}}^{\frac{5}{2}} \int_{-\sqrt{\frac{25}{4} - x^2}}^{\sqrt{\frac{25}{4} - x^2}} \int_{3x^2+3y^2-16}^{9-x^2-y^2} 1\, dV$$

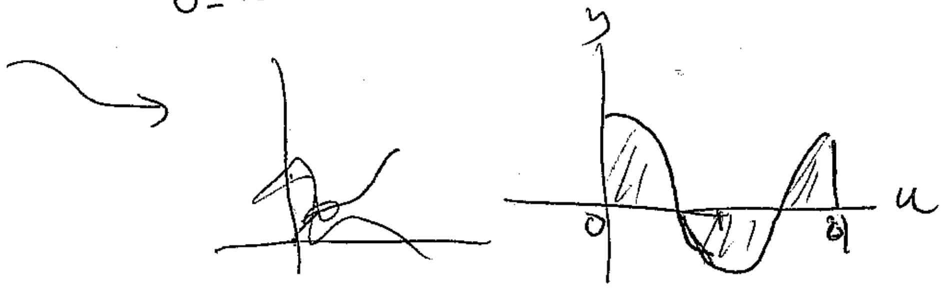
$$= 4 \int_0^{\frac{5}{2}} \int_0^{\sqrt{\frac{25}{4} - x^2}} \int_{3x^2+3y^2-16}^{9-x^2-y^2} 1\, dV.$$

1-D Calc.

$$\int_0^2 2x\cos x^2\,dx = \int_0^4 \cos u\,du = \sin u\Big|_0^4$$

$u = x^2$
$du = 2x\,dx$
$0 \le u \le 4$





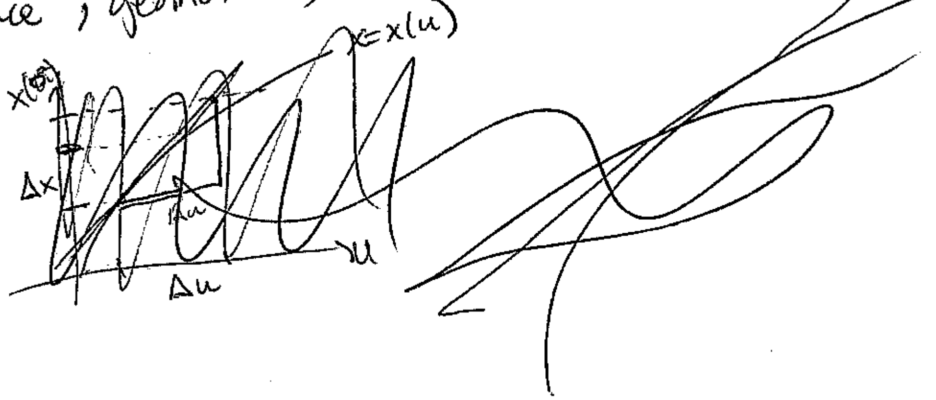
In general

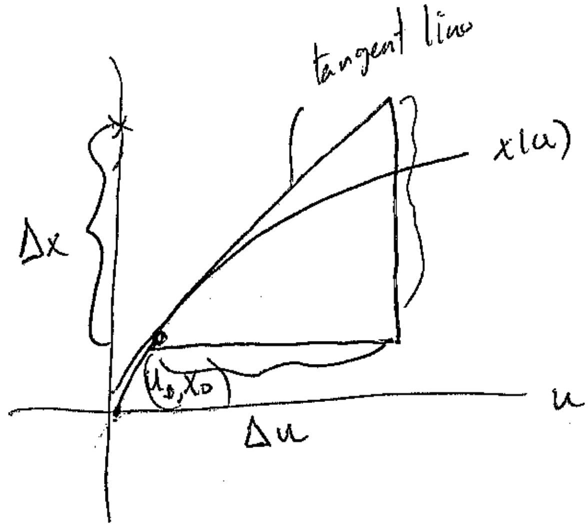
$$\int_A^B f(x)\,dx.$$

Write $x = x(u)$, $x(a) = A$, $x(b) = B$, $dx = x'(u)\,du$

We write $\int_a^b f(x(u)) \cdot x'(u)\,du$.

We're changing the region over which we're integrating and notice, geometrically, that





Slope of tangent line $= x'(u_0)$

and $\dfrac{\Delta x}{\Delta u}$. So $x'(u_0)\Delta u = \Delta x$.

So as $\Delta x \to dx$,
$x'(u_0)\Delta u \to x'(u)\,du$

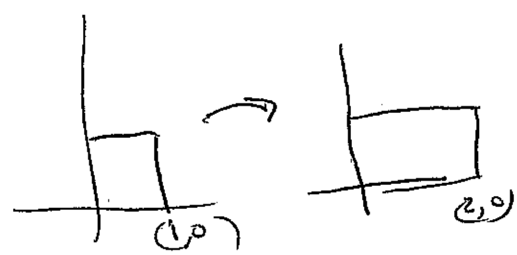
Change of variables for double integrals.

Def. ~~The~~ Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ distort the plane by $T(u,v) = (x(u,v), y(u,v))$.

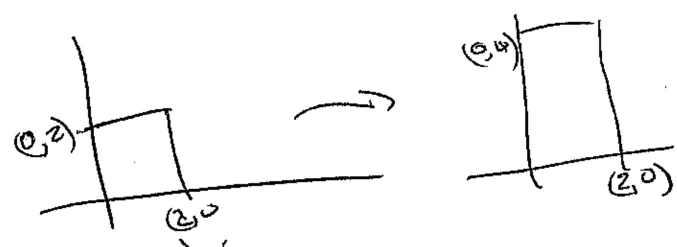
The Jacobian of T $\dfrac{\partial(x,y)}{\partial(u,v)}$ is the determinant of $DT(u,v)$.

e.g. of plane-distorting T:

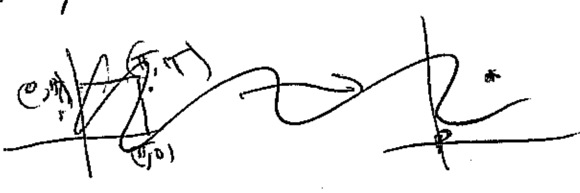
$T(u,v) = (2u, v)$ sends



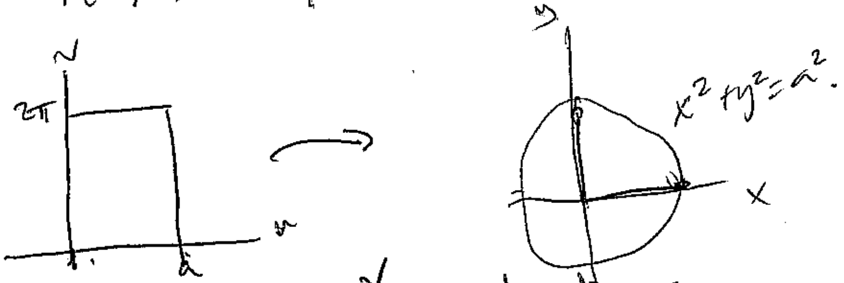
$T(u,v) = (u, v^2)$ sends,



~~$T(u,v) = (u\cos v, u \sin v)$~~



$T(u,v) = (u\cos v, u\sin v)$



$T(u,v)$ sends v-axis to the origin

$T(u,v)$ sends u-axis to $[0,a]$ on x axis

sends $v=\frac{\pi}{2}$ to $[0,a]$ on y-axis

for $0<v<\frac{\pi}{2}$ it sends v $[0,a]$ to $[0,a]$ at angle a.

So you get the circle $x^2+y^2=a^2$

<u>Thm</u> Let D^* be an elementary region in uv-plane
D " in xy-plane

$T: D^* \to D$ is onto and diff'ble. Then we can substitute as follows...

$$\iint_D f(x,y)\,dx\,dy = \iint_{D^*} f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

<u>e.g</u> Let D be the disk of radius a and f be integrable on D.

Then

$$\iint_D f(x,y)\,dx\,dy = \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} f(x,y)\,dx\,dy.$$

Let's substitute with polar coordinates.

$x = r\cos\theta$

$y = r\sin\theta$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \det(DT(r,\theta)) = \det\begin{bmatrix}\cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta\end{bmatrix}$$

~~$T(x,y)$~~ $(x,y) = T(r,\theta) = (r\cos\theta, r\sin\theta)$

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r.$$

So, by thm,

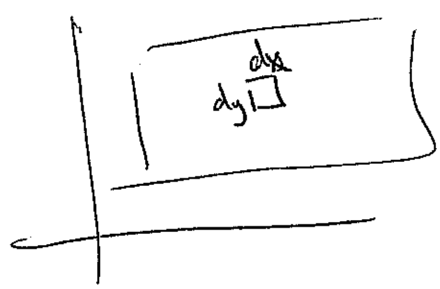
$$\iint_D f(x,y)\,dx\,dy = \iint_{D^*} f(x(r,\theta), y(r,\theta))\, r\, d\theta\, dr$$

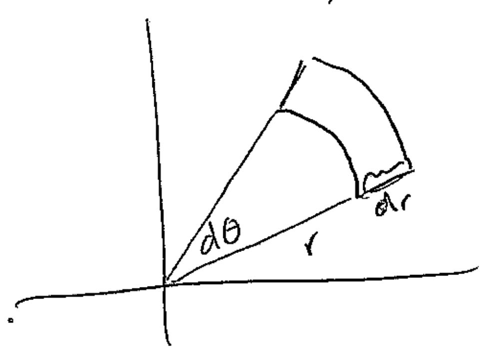
Since T sends circles $x^2+y^2=a^2$ to the rectangle $[0,a]\times[0,2\pi)$

we get

$$\cancel{\iint_{D^*} f} \quad \boxed{\iint_{D^*} f(x(r,\theta), y(r,\theta))\, r\, d\theta = \int_0^{2\pi}\int_0^a f(x(r,\theta), y(r,\theta))\, r\, dr\, d\theta.}$$

On the xy-plane a tiny piece of area is $dx\,dy$





How big is this little slice

as $d\theta \to 0$ it becomes a rectangle of size $\underbrace{dr}^{\text{length}} \times \underbrace{r\,d\theta}^{\text{length}}$

since θ is measured in radians