Major Facts about Directional Derivatives

Fact 1. Let $f : X \subset \mathbb{R}^n \to \mathbb{R}$ be a function that is differentiable at $a \in X$. **Then** the directional derivative $D_{\mathbf{u}} f(\mathbf{a})$ exists for all unit vectors $\mathbf{u} \in \mathbb{R}^n$ (that is, for all directions) and $D_{\mathbf{u}} f(\mathbf{a}) = \nabla f(\mathbf{a}) \cdot \mathbf{u}$.

Fact 2. For a fixed $\mathbf{a} \in \mathbb{R}^n$, the directional derivative $D_{\mathbf{u}} f(\mathbf{a})$ is maximized when $\mathbf{u}$ points in the **same** direction as $\nabla f(\mathbf{a})$ and is minimized when $\mathbf{u}$ points in the **opposite** direction. The maximal and minimal values of $D_{\mathbf{u}} f(\mathbf{a})$ are $||\nabla f(\mathbf{a})||$ and $-||\nabla f(\mathbf{a})||$, respectively.

Fact 3. Let $f : X \subset \mathbb{R}^n \to \mathbb{R}$ be a differentiable function that has all its partial derivatives continuous, $S = \{\mathbf{x} \in X \mid f(\mathbf{x}) = c\}$ be a level set for f, and $\mathbf{a} \in S$. **Then** $\nabla f(\mathbf{a})$ is perpendicular to S.

Fact 4. Let $S \subset \mathbb{R}^3$ be a surface defined by an equation $f(x, y, z) = c$ and $\mathbf{x}_0 \in S$. Assume that f is differentiable at $\mathbf{x}_0$ and has all its partial derivatives continuous. **Then** the tangent plane to S at $\mathbf{x}_0$ is given by the equation $\nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0) = 0$.