Major Facts about Derivatives

Fact 1. If $\mathbf{F} : X \subset \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $\mathbf{a} \in X$, **then** it is continuous at $\mathbf{a}$.

Fact 2. Let $\mathbf{F} : X \subset \mathbb{R}^n \to \mathbb{R}^m$ such that all $\dfrac{\partial F_i}{\partial x_j}$ exist and are continuous in a neighborhood of $\mathbf{a} \in X$. **Then** $\mathbf{F}$ is differentiable at $\mathbf{a}$.

Fact 3. $\mathbf{F} : X \subset \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $\mathbf{a} \in X$ **if and only if** all $F_i : X \subset \mathbb{R}^n \to \mathbb{R}$ are differentiable at $\mathbf{a}$, **where** $\mathbf{F} = (F_1, F_2, \ldots, F_m)$.

Fact 4. **(Linearity of the derivative)**

Let $\mathbf{F}, \mathbf{G} : X \subset \mathbb{R}^n \to \mathbb{R}^m$ be differentiable at $\mathbf{a} \in X$ and $k \in \mathbb{R}$. **Then**

1. $\mathbf{F} + \mathbf{G}$ is differentiable at $\mathbf{a}$ and $D(\mathbf{F} + \mathbf{G})(\mathbf{a}) = D\mathbf{F}(\mathbf{a}) + D\mathbf{G}(\mathbf{a})$.
2. $k\mathbf{F}$ is differentiable at $\mathbf{a}$ and $D(k\mathbf{F})(\mathbf{a}) = kD\mathbf{F}(\mathbf{a})$.

Fact 5. Let $f, g : X \subset \mathbb{R}^n \to \mathbb{R}$ be differentiable at $\mathbf{a} \in X$. **Then**

1. fg is differentiable at $\mathbf{a}$ and $D(fg)(\mathbf{a}) = g(\mathbf{a})Df(\mathbf{a}) + f(\mathbf{a})Dg(\mathbf{a})$.
2. **if** $g \neq 0$, f/g is differentiable at $\mathbf{a}$ and $D(f/g)(\mathbf{a}) = \dfrac{g(\mathbf{a})Df(\mathbf{a}) - f(\mathbf{a})Dg(\mathbf{a})}{g(\mathbf{a})^2}$.

Fact 6. **(The chain rule)** If $\mathbf{F} : X \subset \mathbb{R}^n \to \mathbb{R}^m$ and $\mathbf{G} : Y \subset \mathbb{R}^m \to \mathbb{R}^p$ are differentiable at $\mathbf{a}$ and $\mathbf{b} = \mathbf{F}(\mathbf{a})$, respectively, and $\text{Range}(\mathbf{F}) \subset Y$, **then** $\mathbf{G} \circ \mathbf{F} : X \subset \mathbb{R}^n \to \mathbb{R}^p$ is also differentiable at $\mathbf{a}$ and $D(\mathbf{G} \circ \mathbf{F})(\mathbf{a}) = D\mathbf{G}(\mathbf{b})D\mathbf{F}(\mathbf{a})$.