Chapter 28

Discrete group actions

In the previous chapter, we surveyed hyperbolic geometry. Ultimately, our goal in this part is to understand spaces that locally look like the hyperbolic plane but have some interesting global structure, coming from arithmetic. In order to get off the ground, in this chapter we consider a general context for nice group actions on topological spaces, indicating how these fit in to more general notions in topology. Pathologies exist! Our goal in this chapter is to provide basic context (for further see references in Paragraph 28.7.1) before turning to the central case of interest: a discrete group acting properly on a locally compact, Hausdorff topological space.

[[More introduction and motivation?]]

28.1 Topological group actions

Group actions will figure prominently in what follows, so we set a bit of notation. Our groups will act on the left and on the right; if not specified, we assume a left action, and let G act on X via

$$\begin{aligned} G \times X &\to X \\ (g, x) &\mapsto gx. \end{aligned}$$

We will also sometimes write $G \circlearrowright X$ for an action of G on X.

Example 28.1.1. A group G acts on itself by left multiplication, the *(left) regular group action* of G. If $H \leq G$ is a subgroup, then H also acts on G by left multiplication. For example, if V is an $\mathbb{R}$-vector space with $\dim_{\mathbb{R}} V = n$, and $\Lambda \subseteq V$ is a (full) $\mathbb{Z}$-lattice in V, then $L \cong \mathbb{Z}^n$ is a group and L acts on V by translation.

Another important and related example is the left action of G on the set of right cosets $X = G/H$ again by multiplication, namely

$$g(xH) = gxH \quad \text{for } g \in G \text{ and } xH \in G/H.$$

Let G act on X. The *G-orbit* of $x \in X$ is

$$Gx = \{gx : g \in G\}.$$

The set of G-orbits forms the *quotient set*

$$G\backslash X = \{Gx : x \in X\},$$

with a natural surjective *quotient map*

$$\pi : X \to G\backslash X.$$

Remark 28.1.2. We write $G\backslash X$ for the quotient, as G acts on the left; soon, X will also have a right action and successfully comparing the two will require keeping these sorted.

Example 28.1.3. A group G acts transitively on a nonempty set X if and only if $G\backslash X$ is a single point. In particular, if $H \leq G$, then the action of G on G/H is transitive.

If $H \leq G$, then the quotient set $H\backslash G$ is the set of left cosets of H in G. For example, if $\Lambda = \mathbb{Z}^n \leq \mathbb{R}^n = V$, then $\Lambda\backslash V \simeq [0,1)^n$.

For $x \in X$, we define the *stabilizer* of x by

$$\mathrm{Stab}_G(x) = \{g \in G : gx = x\}.$$

Definition 28.1.4. The action of G on X is *free* (and we say G acts *freely* on X) if $\mathrm{Stab}_G(x) = \{1\}$ for all $x \in X$, i.e., $gx = x$ implies $g = 1$ for all $x \in X$.

Definition 28.1.5. Let X', X be sets with an action of G. A map $f : X' \to X$ is *G-equivariant* if $f(gx') = g(f(x'))$ for all $x' \in X'$ and $g \in G$, i.e., the following diagram commutes:

$$\begin{tikzcd} X' \arrow[r, "f"] \arrow[d, "g"] & X \arrow[d, "g"] \\ X' \arrow[r, "f"] & X \end{tikzcd}$$

If $f : X' \to X$ is G-equivariant, then f induces a map

$$\begin{aligned} G\backslash X' &\to G\backslash X \\ Gx' &\mapsto Gf(x'), \end{aligned}$$

well-defined by the G-equivariance of f, and the following diagram commutes:

$$\begin{tikzcd} X' \arrow[r, "f"] \arrow[d, "\pi"] & X \arrow[d, "\pi'"] \\ G\backslash X' \arrow[r] & G\backslash X \end{tikzcd}$$

Now topology enters. Let G be a *topological group*, a group with a topology in which the multiplication and inversion maps are continuous. Let X be a topological space, and let G act on X. We want to consider only those actions in which the topology on G and on X are compatible.

Definition 28.1.6. The action of G on X is *continuous* if the map $G \times X \to X$ is continuous.

Example 28.1.7. The left regular action of a group on itself is continuous—indeed, together with continuity of inversion (and existence of the identity), this is the definition of a topological group.

Lemma 28.1.8. *Suppose G has the discrete topology. Then an action of G on X is continuous if and only if for all $g \in G$ the left-multiplication map*

$$\begin{aligned} \lambda_g : X &\to X \\ x &\mapsto gx \end{aligned}$$

is continuous; and when this holds, each λ_g is a homeomorphism.

Proof. Exercise 28.2. □

From now on, suppose G acts continuously on X; more generally, whenever G is a topological group acting on a topological space X, we will implicitly assume that the action is continuous.

28.1.9. The quotient $G\backslash X$ is equipped with the *quotient topology*, so that the quotient map $\pi : X \to G\backslash X$ is continuous: a subset $V \subseteq G\backslash X$ is open if and only if $\pi^{-1}(V) \subseteq X$ is open.

The projection π is an open map, which is to say if $U \subseteq X$ is open then $\pi(U) = GU \subseteq G\backslash X$ is open: indeed, if U is open then $\pi^{-1}(\pi(U)) = \bigcup_{g \in G} gU$ is open, so $\pi(U)$ is open by definition of the topology.

28.1.10. If G acts continuously on X, then the topologies on G and X are related by this action. In particular, for any $x \in X$, the natural map

$$\begin{aligned} G &\to Gx \subseteq X \\ g &\mapsto gx \end{aligned}$$

is continuous (it is the restriction of the action map to $G \times \{x\}$). This map factors naturally as

$$G/\operatorname{Stab}_G(x) \to Gx \tag{28.1.11}$$

where we give $G/\operatorname{Stab}_G(x)$ the quotient topology, and this is then a bijective continuous map. The map (28.1.11) need not always be a homeomorphism (Exercise 28.5), but we will see below that it becomes a homeomorphism under further nice hypotheses (Proposition 28.3.11).

28.2 Covering space and wandering actions

Throughout, let X be a Hausdorff topological space with an action of a Hausdorff topological group G. The nicest action of G on X would be one for which the quotient map $\pi : X \to G\backslash X$ is a covering space map and the quotient $G\backslash X$ is also Hausdorff. Neither of these two conditions implies the other. We begin with the first.

Definition 28.2.1. We say the action of G on X is a *covering space action* if for all $x \in X$, there exists an open neighborhood $U \ni x$ such that $gU \cap U \neq \emptyset$ for all $g \in G$ with $g \neq 1$.

28.2.2. If the action of G is a covering space action, then the quotient map $\pi : X \to G\backslash X$ is a *local homeomorphism*, i.e., for every $x \in X$, there exists an open neighborhood $U \ni x$ such that $\pi|_U : U \to \pi(U) \subseteq X$ is a homeomorphism. A local homeomorphism need not conversely be a covering space map.

If G acts by a covering space action, then G acts freely on X. This is too strong a hypothesis on the group actions we will consider in the rest of this book, so we need to look for something weaker. So we consider the following.

Definition 28.2.3. We say that the action of G is *wandering* if for all $x \in X$, there exists an open neighborhood $U \ni x$ such that $gU \cap U \neq \emptyset$ for all but finitely many $g \in G$.

Example 28.2.4. If G is a finite group, then any (continuous) action of G is wandering.

28.2.5. If the action of G is wandering, then immediately we see that for any $x \in X$, the orbit $Gx \subseteq X$ is closed and discrete.

Wandering actions generalize covering space actions, and can be equivalently characterized, as follows.

Lemma 28.2.6. *The following are equivalent:*

(i) *The action of G is wandering; and*

(ii) *For all $x \in X$, we have $\#\operatorname{Stab}_G(x) < \infty$, and there exists an open neighborhood $U \ni x$ such that $gU \cap U \neq \emptyset$ implies $g \in \operatorname{Stab}_G(x)$.*

If G acts freely, then these are further equivalent to:

(iii) *The action of G is by a covering space action.*

Proof. The implication (ii) $\Rightarrow$ (i) is immediate; we prove the converse. Let U be a neighborhood of $x \in X$ such that $gU \cap U \neq \emptyset$ for only finitely many $g \in G$. We have $\#\operatorname{Stab}_G(x) < \infty$ since $g \in \operatorname{Stab}_G(x)$ implies $x \in gU \cap U$. Let

$$\{g \in G : gU \cap U \neq \emptyset \text{ and } gx \neq x\} = \{g_1, \ldots, g_n\}.$$

Since X is Hausdorff, for all i there exist open neighborhoods $V_i, W_i \subseteq X$ of $x, g_i x$, respectively, such that $V_i \cap W_i = \emptyset$. Since G acts continuously, there exists an open neighborhood $W_i' \subseteq X$ of x such that $g_i W_i' \subseteq W_i$. Let $U_i = V_i \cap W_i'$. Then $x \in U_i$ and

$$U_i \cap g_i U_i \subseteq U_i \cap g W_i' \subseteq V_i \cap W_i = \emptyset.$$

Then $U' = \bigcap_i U_i$ has the desired property in (ii): if $gU' \cap U' \neq \emptyset$ then either $gx = x$ or $g = g_i$ for some i, and $g_i U' \cap U' \subseteq g_i U_i \cap U_i = \emptyset$.

Finally, if G acts freely, then $\operatorname{Stab}_G(x)$ is trivial for all x; this shows that (ii) $\Leftrightarrow$ (iii). $\square$

28.2.7. Suppose the action of G is wandering. Then at a point x with neighborhood $U \ni x$ and finite stabilizer $\mathrm{Stab}_G(x)$, we can replace U by $\bigcap_{g \in \mathrm{Stab}_G(x)} gU$ so that $U \ni x$ is an open neighborhood on which G acts. Then the projection map factors

$$\pi|_U : U \to \mathrm{Stab}_G(x)\backslash U \to G\backslash X$$

and the latter map is a homeomorphism onto its image; we say π is a *local homeomorphism modulo stabilizers*. If G is free, then we recover Paragraph 28.2.2.

Remark 28.2.8. If G has the discrete topology and the condition in Lemma 28.2.6(ii) holds, then some authors call the action of G *properly discontinuous*. This is probably because G is then as broken ("discontinuous") as possible: G has the discrete topology, and we should be able to find neighborhoods that pull apart the action of G. This nomenclature is strange because we still want the action to be *continuous* (just by a discrete group). Adding to the potential confusion is the issue that different authors give different definitions of "properly discontinuous" depending on their purposes; most of these can be seen to be equivalent under the right hypotheses on the space, but not all. We avoid this term.

It turns out that a wandering action is too weak a property in this level of generality for us to work with. However, it is close, and we will shortly see that it suffices with additional hypotheses on the space X.

28.3 Hausdorff quotients and proper group actions

In this section we define proper group actions; to motivate this definition, we first ask for conditions that imply that the quotient $G\backslash X$ is Hausdorff.

Lemma 28.3.1. *The following are equivalent.*

(i) *The quotient $G\backslash X$ is Hausdorff;*

(ii) *If $Gx \neq Gy \in G\backslash X$, then there exist open neighborhoods $U \ni x$ and $V \ni y$ such that $gU \cap V = \emptyset$ for all $g \in G$; and*

(iii) *For all $x, y \in X$, there exist open neighborhoods $U \ni x$ and $V \ni y$ such that for all $g \in G$, we have $gU \cap V \neq \emptyset$ if and only if $gx = y$; and*

(iv) *The image of the action map*

$$\begin{aligned} G \times X &\to X \times X \\ (g, x) &\mapsto (x, gx) \end{aligned} \tag{28.3.2}$$

is closed.

Proof. The implication (i) $\Leftrightarrow$ (ii) follows directly from properties of the quotient map: the pullback of open neighborhoods separating Gx and Gy under the continuous projection map have the desired properties, and conversely the pushforward of the given neighborhoods under the open projection map separate Gx and Gy. The implication (ii) $\Rightarrow$ (iii) is immediate.

To conclude, we prove (i) $\Leftrightarrow$ (iv). We use the criterion that a topological space is Hausdorff if and only if the diagonal map has closed image. The continuous surjective map $\pi : X \to G\backslash X$ is open, so the same is true for

$$\pi \times \pi : X \times X \to (G\backslash X) \times (G\backslash X).$$

Therefore the diagonal is closed in $(G\backslash X) \times (G\backslash X)$ if and only if its preimage is closed in $X \times X$. But this preimage consists exactly of the orbit relation

$$\{(x, x') \in X \times X : x' = gx \text{ for some } g \in G\},$$

and this is precisely the image of the action map (28.3.2). □

The conditions Lemma 28.3.1(i)–(iii) can sometimes be hard to verify, so it is convenient to have a condition that implies Lemma 28.3.1(iv); this definition will seek to generalize the situation when G is compact. First, we make a definition.

Definition 28.3.3. Let $f : X \to Y$ be a continuous map.

(a) We say $f : X \to Y$ is *quasi-proper* if $f^{-1}(K)$ is compact for all compact $K \subseteq Y$.

(b) We say f is *proper* if f is quasi-proper and closed (the image of every closed subset is closed).

Example 28.3.4. If X is compact, then any continuous map $f : X \to Y$ is proper because f is closed and if $K \subseteq Y$ is compact, then K is closed, so $f^{-1}(K) \subset X$ is closed hence compact, since X is compact.

28.3.5. Suppose that Y is locally compact and Hausdorff. If $f : X \to Y$ is continuous, then f is quasi-proper if and only if f is proper (Exercise 28.7), and in either case X is locally compact: cover Y with open relatively compact sets $U_i \subseteq K_i$; then $V_i = f^{-1}(U_i)$ is an open cover of X by relatively compact sets.

Remark 28.3.6. There is an alternate characterization of proper maps as follows: a continuous map $f : X \to Y$ is proper if and only if the map $f \times \mathrm{id} : X \times Z \to Y \times Z$ is closed for every topological space Z. See Paragraph 28.7.1 for more discussion.

Partly motivated by Lemma 28.3.1(iv), we make the following definition.

Definition 28.3.7. The action of G on X is *proper* (G acts *properly* on X) if the *action map*

$$\begin{aligned} \lambda : G \times X &\to X \times X \\ (g, x) &\mapsto (x, gx) \end{aligned} \tag{28.3.8}$$

is proper.

Proposition 28.3.9. *If G is compact, then any (continuous) action of G on (a Hausdorff space) X is proper.*

Proof. Let $K \subseteq X \times X$ be compact; then K is closed (because X is Hausdorff). Let K_1 be the projection of K onto the first factor. Then K_1 is compact, and $\lambda^{-1}(K)$ is a closed subset of the compact set $G \times K_1$, so it is compact. This shows that the action map is quasi-proper. Finally, the action map is closed because it factors

$$\begin{aligned} G \times X &\to G \times X \times X \to X \times X \\ (g, x) &\mapsto \ (g, x, gx) \ \mapsto (x, gx); \end{aligned}$$

the first map is a homeomorphism, and the second (projection) map is closed, as G is compact (by a standard application of the tube lemma). □

Example 28.3.10. If G is a finite discrete group, then G acts properly by Proposition 28.3.9.

Proper actions have many of the properties we need.

Proposition 28.3.11. *Let G act properly on X. Then the following are true.*

(a) *$G\backslash X$ is Hausdorff.*

(b) *The orbit $Gx \subseteq X$ is closed for all $x \in X$.*

(c) *The group $\mathrm{Stab}_G(x)$ is compact for all $x \in X$.*

(d) *The natural map $\iota_x : G/\mathrm{Stab}_G(x) \to Gx$ is a homeomorphism.*

Proof. For part (a), by Lemma 28.3.1, it is enough to note that by definition the image of the action map λ in (28.3.8) is closed. Part (b) follows in the same way, as

$$Gx \simeq \{x\} \times Gx = \lambda(G \times \{x\}).$$

For (c), let $\lambda : G \times X \to X \times X$ be the action map and let $x \in X$. Then we have by definition that $\lambda^{-1}(x, x) = \mathrm{Stab}_G(x) \times \{x\} \simeq \mathrm{Stab}_G(x)$, so by definition $\mathrm{Stab}_G(x)$ is compact.

Finally, part (d) (cf. Paragraph 28.1.10). The map ι_x is bijective and continuous; we claim it is also closed, whence a homeomorphism. The natural map $G \to Gx$ is closed for the same reason as in part (b), and the map ι_x is a factor of this map via the surjection $G \to G/\mathrm{Stab}_G(x)$, so ι_x is also closed. □

When X is locally compact, our central case of interest, then proper maps can be characterized in a way that compares to a wandering action as follows. Recall our running assumption that X and G are Hausdorff.

Theorem 28.3.12. *Suppose that X is locally compact and let G act on X. Then the following are equivalent.*

(i) *G is discrete and acts properly on X;*

(ii) *For all compact subsets $K \subseteq X$, we have $K \cap gK \neq \emptyset$ for only finitely many $g \in G$;*

(iii) *For all compact subsets $K, L \subseteq X$, we have $K \cap gL \neq \emptyset$ for only finitely many $g \in G$; and*

(iv) *For all $x, y \in X$, there exist open neighborhoods $U \ni x$ and $V \ni y$ such that $U \cap gV \neq \emptyset$ for only finitely many $g \in G$.*

Moreover, if X is a locally compact metric space with G acting by isometries, then these are further equivalent to:

(v) *The action of G on X is wandering; and*

(vi) *For all $x \in X$, the orbit $Gx \subseteq X$ is discrete and $\#\operatorname{Stab}_G(x) < \infty$.*

Proof. First, we show (i) $\Rightarrow$ (ii). Let $\lambda : G \times X \to X \times X$ be the action map. Let $K \subseteq X$ be compact. Then

$$\lambda^{-1}(K \times K) = \{(g, x) \in G \times X : x \in K, gx \in K\}$$

is compact by definition. The projection of $\lambda^{-1}(K \times K)$ onto G is compact, and since G is discrete, this projection is finite and includes all $g \in G$ such that $K \cap gK \neq \emptyset$.

Next we show (ii) $\Leftrightarrow$ (iii): The implication (ii) $\Leftarrow$ (iii) is immediate, and conversely we apply (ii) to the compact set $K \cup L$ to conclude

$$K \cap gL \subseteq (K \cup L) \cap g(K \cup L) \neq \emptyset$$

for only finitely many $g \in G$.

Next we show (ii) $\Rightarrow$ (iv). For all $x \in X$, since X is locally compact there is a compact neighborhood $K \supset U \ni x$, with U open and K compact. If $U \cap gU \neq \emptyset$ then $K \cap gK \neq \emptyset$ and this happens for only finitely many $g \in G$.

Finally, we show (iv) $\Rightarrow$ (i). We first show that the action map is quasi-proper, and conclude that it is proper by Paragraph 28.3.5. Let $K \subseteq X \times X$ be compact. By (iv), for any $(x, y) \in K$, there exist neighborhoods $U \ni x$ and $V \ni y$ such that the set

$$W = \{g \in G : gU \cap V \neq \emptyset\}$$

is finite. The set $U \times V \ni (x, y)$ is an open neighborhood of $(x, y) \in K$, and so the collection of these neighborhoods ranging over $(x, y) \in K$ is an open cover of K, so finitely many $U_i \times V_i \ni (x_i, y_i)$ suffice, and with corresponding sets $\#W_i < \infty$. Let $W = \bigcup_i W_i \subseteq G$. Let $K_1 \subseteq X$ be the projection of K onto the first coordinate. We claim that $\lambda^{-1}(K) \subseteq W \times K_1$: indeed, if $\lambda(g, x) = (x, gx) \in K$ then $x \in K_1$ and $(x, gx) \in U_i \times V_i$ for some i, so $gx \in gU_i \cap V_i$ so $g \in W_i$, and $(g, x) \in W \times K_1$. Since $\#W < \infty$ and K_2 is compact, $W \times K_2$ is compact; since K is compact, K is closed so $\lambda^{-1}(K) \subseteq W \times K_2$ is also closed hence compact.

To conclude that G is discrete, we argue as follows. For all $x \in X$, the orbit $Gx \subseteq X$ is discrete: taking $U = V$ and a neighborhood $U \ni x$ with $U \cap gU \neq \emptyset$ for only finitely many $g \in G$, we see that $U \cap Gx$ is finite so Gx is discrete (as X is Hausdorff). By Proposition 28.3.11(d), the map $G/\operatorname{Stab}_G(x) \to Gx$ is a homeomorphism for any $x \in X$. Since $\#\operatorname{Stab}_G(x) < \infty$, this implies that G is discrete: given $g \in G$, we

find an open neighborhood $U \ni g$ with $U \cap G \subseteq \mathrm{Stab}_G(x)$ finite Hausdorff, hence discrete. This completes the equivalence (i)–(iv).

Next, we show (ii) $\Rightarrow$ (v) in all cases. Let $x \in X$ and let $K \supseteq U \ni x$ be a compact neighborhood. By (ii), $K \cap gK \neq \emptyset$ for only finitely many $g \in G$. Thus $U \cap gU \neq \emptyset$ for only finitely many $g \in G$, and so G is wandering, proving (v). The implication (v) $\Rightarrow$ (vi) also holds in all cases and is immediate (Proposition 28.2.5).

Finally, we show (vi) $\Rightarrow$ (ii) under the extra hypothesis that X is a metric space with G acting by isometries. Assume for purposes of contradiction that there exist infinitely many $g_n \in G$ such that $K \cap g_n K \neq \emptyset$, and accordingly let $x_n \in K$ with $g_n x_n \in K$. The points x_n accumulate in K, so we may assume $x_n \to x \in K$; by taking a further subsequence, we may assume $g_n x_n \to y \in K$. We then claim that the set $\{g_n x\}_n$ accumulates near y. Since $\#\,\mathrm{Stab}_G(x) < \infty$, we may assume without loss of generality that the points $g_n x$ are all distinct. Then, given $\epsilon > 0$, we have

$$d(g_n x, y) \leq d(g_n x, g_n x_n) + d(g_n x_n, y) < d(x, x_n) + d(g_n x_n, y) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

for n sufficiently large. This contradicts that the orbit Gx is discrete, having no limit points. □

Remark 28.3.13. The hypothesis "X is a metric space with G acting by isometries" providing the equivalent condition Theorem 28.3.12(v) is necessary: see Exercise 28.10.

28.3.14. From Lemma 28.2.6 and the implication Theorem 28.3.12 (v) $\Rightarrow$ (i), we see that proper actions generalize covering space actions when X is locally compact metric space and G acts by isometries. In fact, a more general statement is true: if G is a discrete group with a covering space action on X such that $G\backslash X$ is Hausdorff, then G acts properly on X. The (slightly involved) proof in general is requested in Exercise 28.14.

28.4 Symmetric space model

In this section, before proceeding with our treatment of discrete group actions in our case of interest, we pause to give a very important way to think about hyperbolic space in terms of symmetric spaces. The magical formulas in hyperbolic geometry beg for a more conceptual explanation: what is their provenance? Although it is important for geometric intuition to begin with a concrete model of hyperbolic space and asking about its isometries directly, from this point of view it is more natural to instead *start* with the desired group and have it act on itself in a natural way.

28.4.1. Let $G = \mathrm{SL}_2(\mathbb{R})$. As a matrix group, G comes with a natural metric. The space $\mathrm{M}_2(\mathbb{R}) \simeq \mathbb{R}^4$ has the usual structure of a metric space, with

$$\|g\|^2 = a^2 + b^2 + c^2 + d^2, \quad \text{if } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{M}_2(\mathbb{R}).$$

We give $\mathrm{SL}_2(\mathbb{R}) \subset \mathrm{M}_2(\mathbb{R})$ the subspace metric and $\mathrm{PSL}_2(\mathbb{R})$ the quotient metric. Intuitively, in this metric $g, h \in \mathrm{PSL}_2(\mathbb{R})$ are close if there exist matrices representing g, h (corresponding to a choice of sign) with all four entries of the matrix close in $\mathbb{R}$.

28.4.2. Recall from Paragraph 28.1.10 that if G acts (continuously and) transitively on X, then for any $x \in X$, the natural map $g \mapsto gx$ gives a continuous bijection

$$G/\operatorname{Stab}_G(x) \xrightarrow{\sim} Gx = X.$$

Let $X = \mathbf{H}^2$ be the hyperbolic plane and let $G = \mathrm{SL}_2(\mathbb{R})$. Then G acts transitively on X. The stabilizer of $x = i$ is the subgroup $K = \operatorname{Stab}_G(x) = \mathrm{SO}(2) \leq \mathrm{SL}_2(\mathbb{R})$, so we have a continuous bijection

$$\begin{aligned} G/K = \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}(2) &\xrightarrow{\sim} \mathbf{H}^2 = X \\ gK &\mapsto gi. \end{aligned} \tag{28.4.3}$$

From the Iwasawa decomposition (Proposition 27.3.2), it follows that

$$\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}(2) \simeq NA.$$

In fact, the map (28.4.3) is a homeomorphism. To prove this, we observe the following beautiful equation: for $g \in \mathrm{SL}_2(\mathbb{R})$, we have

$$\|g\|^2 = 2\cosh \rho(i, gi). \tag{28.4.4}$$

This follows from the formula (27.4.3) for distance; the calculation is requested in Exercise 28.15. It follows that the map $G \to X$ is open, and thus (28.4.3) is a homeomorphism. In fact, by (28.4.4), if we reparametrize the metric on $\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}(2)$ or on $\mathbf{H}^2$ by $\cosh$, the map 28.4.3 becomes an isometry.

28.4.5. Let G be a Hausdorff, locally compact topological group. For example, we may take $G = \mathrm{SL}_2(\mathbb{R})$. Then G has a translation invariant Borel measure μ, unique up to scaling, called the *Haar measure*. If $G = \mathbb{R}^n$, then this measure is the usual Lebesgue measure.

Definition 28.4.6. Let G be a Hausdorff, locally compact topological group. A *lattice* $\Gamma \leq G$ in G is a discrete subgroup with cofinite volume, i.e., such that $\mu(\Gamma\backslash G) < \infty$.

28.5 Fuchsian groups

We now specialize to our case of interest and consider the group $\mathrm{PSL}_2(\mathbb{R})$ acting by isometries on the geodesic space $\mathbf{H}^2$.

Lemma 28.5.1. *Let $\Gamma \leq \mathrm{SL}_2(\mathbb{R})$. Then the following are equivalent.*

(i) *Γ is discrete;*

(ii) *If $\gamma_n \in \Gamma$ and $\gamma_n \to 1$, then $\gamma_n = 1$ for almost all n; and*

(iii) *For all $M \in \mathbb{R}_{>0}$, the set $\{\gamma \in \Gamma : \|\gamma\| \leq M\}$ is finite.*

Proof. The equivalence (i) $\Leftrightarrow$ (ii) is requested in Exercise 28.13. The implication (i) $\Leftrightarrow$ (iii) follows from the fact that the ball of radius M in $\mathrm{SL}_2(\mathbb{C})$ a compact subset of $\mathrm{M}_2(\mathbb{R})$, and subset of a compact set is finite if and only if it is discrete. Slightly more elaborately, a sequence of matrices with bounded norm has a subsequence where the entries all converge; since the determinant is continuous, the limit exists in $\mathrm{SL}_2(\mathbb{R})$ so Γ is not discrete. $\square$

In particular, we find from Lemma 28.5.1 that a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$ is countable.

Proposition 28.5.2. *Let $\Gamma \leq \mathrm{PSL}_2(\mathbb{R})$ be a subgroup with the subspace topology. Then Γ has a wandering action on H^2 if and only Γ is discrete.*

In particular, if Γ is discrete, then Γ acts properly on H^2, by Theorem 28.3.12.

Proof. The implication $\Rightarrow$ is a consequence of Theorem 28.3.12(v) $\Rightarrow$ (i). Conversely, suppose that Γ is discrete; we show that Theorem 28.3.12(vi) holds.

First we show that the stabilizer of a point is finite. We may work in the unit disc D^2 and take the point to be $w = 0 \in \mathsf{D}^2$. The stabilizer of $w = 0$ in $\mathrm{SU}(1,1)$ is $\mathrm{SO}(2) \cong \mathbb{R}/(2\pi)\mathbb{Z}$, so its stabilizer in Γ is a discrete subgroup of $\mathrm{SO}(2)$, so it is necessarily finite (indeed, cyclic).

Next we show that orbits of Γ on H^2 are discrete. We apply the identity (28.4.4). The identity with Lemma 28.5.1 shows that the orbit Γi is discrete. But for any $z \in \mathsf{H}^2$, there exists $h \in \mathrm{PSL}_2(\mathbb{R})$ such that $hi = z$, and conjugation by h induces an isomorphism $\Gamma \xrightarrow{\sim} h^{-1}\Gamma h$ of topological groups. Since

$$\rho(z, gz) = \rho(hi, ghi) = \rho(i, (h^{-1}gh)i)$$

the same argument shows that the orbit Γz is discrete. This concludes the proof.

Alternatively, here is a self-contained proof of the previous paragraph. Since Γ is discrete, there is an ϵ-neighborhood $U \ni 1 \in \mathrm{PSL}_2(\mathbb{R})$ such that $U \cap \Gamma = \{1\}$; therefore, if

$$\gamma = \begin{pmatrix} u & v \\ \overline{u} & -\overline{v} \end{pmatrix} \in \Gamma \setminus \{1\}$$

then $|v| > \epsilon$ or $|u - 1| > \epsilon$. We claim that in either case we have

$$|\gamma(0)| = \left|\frac{v}{u}\right| > \epsilon.$$

If $|v| > \epsilon$, then since $|u| < 1$ anyway we have immediately $|v/u| > \epsilon$; if $|u - 1| > \epsilon$ then $|u| < 1 - \epsilon$ so $|u|^2 < 1 - \epsilon^2$ and $1/|u|^2 > 1 + \epsilon^2$ so

$$\left|\frac{v}{u}\right|^2 = \frac{1 - |u|^2}{|u|^2} > (1 + \epsilon^2) - 1 = \epsilon^2.$$

(One can also show the other implication directly, as follows. Suppose that Γ is not discrete; then there is a sequence $\gamma_n \in \Gamma \setminus \{1\}$ of elements such that $\gamma_n \to 1$. Therefore, for any $z \in \mathsf{H}^2$, we have $\gamma_n z \to z$ and thus $\gamma_n z = z$ for infinitely many n or every neighborhood of z contains infinitely many points $\gamma_n z$.) $\square$

Definition 28.5.3. A *Fuchsian group* is a discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$.

28.6 A primer on Riemannian manifolds

To conclude, we briefly review what we need from the theory of Riemannian manifolds and their quotients. The simplest kind of Riemannian manifold is a Riemann surface, intuitively a topological space which locally looks the complex plane. The formal definition is built so that one can extend the notion of holomorphic function without difficult from the complex plane to a Riemann surface.

28.6.1. A *(topological) n-manifold* is a Hausdorff topological space X locally homeomorphic to $\mathbb{R}^n$. An *atlas* $\{(\phi_i : U_i \hookrightarrow \mathbb{R}^n)\}_i$ for an n-manifold is an open cover $\{U_i\}_i$ with each ϕ_i a homeomorphism of U_i onto an open subset of $\mathbb{R}^n$.

A *Riemann surface* is a smooth 2-manifold such that

$$\phi_i\phi_j^{-1} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j)$$

is a holomorphic map whenever $U_i \cap U_j \neq \emptyset$.

A *morphism* of Riemann surfaces is a continuous map $f : Y \to X$ of Riemann surfaces with atlases $\{(\phi_i, U_i)\}_i$ of X and $\{(\psi_j, V_j)\}_j$ of Y such that each map

$$\phi_i f \psi_j^{-1} : \psi_j(V_j \cap f^{-1}U_i) \to \mathbb{C}$$

is holomorphic. An *isomorphism* of Riemann surfaces is a morphism that is a homeomorphism.

Example 28.6.2. The field $\mathbb{C}$ of complex numbers is the "original" Riemann surface, and any open subset of $\mathbb{C}$ is a Riemann surface.

The simplest nonplanar example of a Riemann surface is the *Riemann sphere* $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$. The atlas on $\mathbb{P}^1(\mathbb{C})$ is given by the open sets $U_1 = \mathbb{P}^1(\mathbb{C}) - \{\infty\} = \mathbb{C}$ and $U_2 = \mathbb{P}^1(\mathbb{C}) - \{0\}$ and atlas $\phi_1 : U_1 \to \mathbb{C}$ by $\phi_1(z) = z$ and $\phi_2 : U_2 \to \mathbb{C}$ by $\phi_2(z) = 1/z$; the map $\phi_2\phi^{-1} = \mathrm{id}$ is analytic on $\phi_1(U_1 \cap U_2) = \mathbb{C} - \{0\}$. Topologically, the Riemann sphere is the one-point compactification of $\mathbb{C}$, and becomes a sphere by stereographic projection. It is also sometimes written $\mathbb{P}^1(\mathbb{C}) = \widehat{\mathbb{C}}$,

Theorem 28.6.3 (Riemann uniformization theorem)**.** *Every (connected and) simply connected Riemann surface H is isomorphic to either the Riemann sphere $\mathbb{P}^1(\mathbb{C})$, the complex plane $\mathbb{C}$, or the hyperbolic plane.*

The universal cover H of a compact Riemann surface X is simply connected, so by the theory of covering spaces, X is a quotient $X \simeq \Gamma\backslash H$ where Γ is the *fundamental group* of X, a subgroup of isometries of H acting by a covering space action. When $H = \mathbb{P}^1(\mathbb{C})$, the only possible group Γ is trivial. When $H = \mathbb{C}$, by classification one sees that the only Riemann surfaces of the form $X = \mathbb{C}/\Gamma$ are the plane $X = \mathbb{C}$, the punctured plane $\mathbb{C}^\times$, and tori $\mathbb{C}/\Lambda$ where $\Lambda \subset \mathbb{C}$ is a full lattice. Therefore, the interesting Riemann surfaces are the *hyperbolic* Riemann surfaces, those of the form $\Gamma\backslash\mathbf{H}^2$.

28.7 Extensions and further reading

A gentle introduction to the geometry of discrete groups is provided by Beardon [Bea95], with a particular emphasis on Fuchsian groups and their fundamental domains; in the notes at the end of each chapter are further bibliographic pointers.

28.7.1. Bourbaki discusses proper maps [Bou60, Chapter I, §10] and more generally groups acting properly on topological spaces [Bou60, Chapter III, §§1,4]; the definition of proper is equivalent to ours as follows. Let $f : X \to Y$ be continuous, and say f is *Bourbaki proper* to mean that $f \times \mathrm{id} : X \times Z \to Y \times Z$ is closed for every topological space Z. If f is Bourbaki proper, then f is proper [Bou60, Chapter I, §10, Proposition 6]. In the other direction, if f is proper then f is closed and $f^{-1}(y)$ is compact for all $y \in Y$, and this implies that f is Bourbaki proper [Bou60, Chapter I, §10, Theorem 1].

A subgroup of isometries $\Gamma \leq \mathrm{Isom}(\mathbf{H}^2)$ is *elementary* if there is a nonempty Γ-invariant set in $\mathbf{H}^2 \cup \partial\mathbf{H}^2$ that contains at most two points. Equivalently, an elementary group is a cyclic subgroup or a (possibly) dihedral group—in particular, an elementary group is virtually abelian (has a finite index subgroup). The elementary groups are easy to analyze, but their inclusion into theorems about more general Fuchsian groups can cause problems; and so in general we are only interested in non-elementary groups.

Non-elementary Fuchsian groups Γ are categorized by the set of limit points $L(\Gamma) \subseteq \partial\mathbf{H}^2$ of Γz with $z \in \mathbf{H}^2$. If $L(\Gamma) = \partial\mathbf{H}^2$, then Γ is said to be a Fuchsian group *of the first kind*; otherwise Γ is *of the second kind*, and $L(\Gamma)$ is a nowhere-dense perfect subset of $\partial\mathbf{H}^2$, topologically a Cantor set. We will see later that if Γ has finite coarea, then Γ is finitely generated of the first kind.

28.7.2. More generally, a *(globally) symmetric space* is any space of the form G/K where G is a Lie group and K a maximal compact subgroup. Alternatively, it can be defined as a space where every point has a neighborhood where there is an isometry of order 2 fixing the point. The theory of symmetric spaces and the connection to differential geometry and Lie groups is described in the book by Helgason [Hel78].

Exercises

28.1. Let G be a group and X a set. Show that the quotient $G\backslash X$ is defined by a universal mapping property: if $f : X' \to X$ is G-equivariant, then there exists a unique map making the diagram

$$\begin{tikzcd} X' \arrow[r, "f"] \arrow[dr, dashed] & X \arrow[d, "\pi"] \\ & G\backslash X \end{tikzcd}$$

28.2. Prove Lemma 28.1.8: if a discrete group G acts on a topological space X, then an action of G on X is continuous if and only if for all $g \in G$ the map $X \to X$ by $x \mapsto gx$ is continuous (therefore, a homeomorphism).

28.3. Let G be a topological group acting on a Hausdorff topological space X. Show that $G\backslash X$ is T_1 (for every pair of distinct points, there exists an open neighborhood of one point not containing the other) if the orbits of G are closed, i.e., $Gx \subseteq X$ is closed for all $x \in X$.

28.4. Let X be a metric space. Then $\mathrm{Isom}(X)$ has naturally the topology of *pointwise convergence*, as follows. We have an embedding

$$\begin{aligned} \mathrm{Isom}(X) &\to X^X = \prod_{x \in X} X \\ g &\mapsto (g(x))_{x \in X}. \end{aligned}$$

The product X^X has the product topology, and so $\mathrm{Isom}(X)$ (and, indeed, any space of maps from X to X) has an induced subspace topology. A basis of open sets for $\mathrm{Isom}(X)$ in this topology consists of finite intersections of open balls

$$V(g; x, \epsilon) = \{h \in \mathrm{Isom}(X) : \rho(g(x), h(x)) < \epsilon\}.$$

Equip the group $G = \mathrm{Isom}(X)$ with the topology of pointwise convergence.

a) Show that G is a topological group.

b) Show that G acts continuously on X.

28.5. Let $G = \mathbb{Z}$ be given the discrete topology, and let $G \circlearrowright X = \mathbb{R}$ act by $x \mapsto x^n$ for $n \in \mathbb{Z}$ (and taking $x^0 = 1$ for all $x \in X$). Show that this action is continuous, and the map (28.1.11)

$$G/\operatorname{Stab}_G(x) \to Gx$$

is *not* a homeomorphism for all $x \in X$.

28.6. Suppose the action of G on X is free and wandering, and let U be an open set such that $gU \cap U = \emptyset$ for all $g \neq 1$. Show that the map $G \times U \to \pi^{-1}(\pi(U))$ is a homeomorphism and the restriction $\pi : G \times U \to \pi(U) \simeq U$ is a (split) covering map.

28.7. Suppose that X is Hausdorff and Y is Hausdorff and locally compact, and let $f : X \to Y$ be a continuous, quasi-proper map. Show that f is closed, hence proper. *[Hint: for a closed set $W \subseteq X$, consider a sequence $y_n \to y \in f(W)$ contained in a compact neighborhood $K \ni y$; choose preimages $x_n \in f^{-1}(K)$ of each y_n, and use compactness and continuity to show that $y \in f(W)$.]*

28.8. One way to weaken the running hypothesis that X is Hausdorff in this chapter is to instead assume only that X is *locally Hausdorff*: every $x \in X$ has an open neighborhood $U \ni x$ such that U is Hausdorff.

Show that if X is only locally Hausdorff, then Lemma 28.2.6 is false: exhibit a topological space X with a free, continuous action of a finite group G that is not wandering, and in particular such that the quotient $\pi : X \to G\backslash X$ is not a local homeomorphism. *[Hint: Let X be the bug-eyed line and $G \cong \mathbb{Z}/2\mathbb{Z}$ acting by $x \mapsto -x$ on $\mathbb{R}^\times$ and swapping the doubled origin.]*

28.9. Let X be locally compact, let $x \in X$, and let $U \ni x$ be an open neighborhood. Show that there exists an open neighborhood $V \ni x$ such that $K = \mathrm{cl}(V) \subseteq U$ is compact.

28.10. Let $G = \mathbb{Z}$ and let $G \hookrightarrow X = \mathbb{R}^2 - \{(0,0)\}$ act by $n \cdot (x, y) = (2^n x, y/2^n)$. In other words, G is the group of continuous maps $X \to X$ generated by $(x, y) \mapsto (2x, y/2)$.

a) Show that the action of G on X is free and wandering.

b) Show that the quotient $G\backslash X$ is not Hausdorff.

c) Let $K = \{(t, 1-t) : t \in [0,1]\}$. Then K is compact. Show that $K \cap gK \neq \emptyset$ for infinitely many $g \in G$.

(So Theorem 28.3.12(v) holds but (ii) does not, and in particular that the action of G is not proper.)

28.11. Show that a subgroup $\Gamma \leq \mathbb{R}^n$ is discrete if and only if and only if $\Gamma = \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_m$ with $v_1, \ldots, v_m \in \mathbb{R}^n$ linearly independent. [[Hint is probably needed here?]]

28.12. Analogous to (28.4.3), show $\mathbf{S}^n \simeq \mathrm{SO}(n+1)/\mathrm{SO}(n)$, where $\mathbf{S}^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$ is the n-dimensional sphere.

28.13. Show that a subgroup $\Gamma \leq \mathrm{SL}_2(\mathbb{R})$ is discrete if and only if whenever $\gamma_n \in \Gamma$ and $\gamma_n \to 1$ then $\gamma_n = 1$ for almost all n.

28.14. Let G be a discrete group with a covering space action on X such that $G\backslash X$ is Hausdorff. Show that G acts properly on X.

28.15. Show that for $g \in \mathrm{SL}_2(\mathbb{R})$, we have

$$\|g\|^2 = 2\cosh \rho(i, gi)$$

(cf. Paragraph 28.4.1).

28.16. The group $\mathrm{SL}_2(\mathbb{R})$ acts on $\mathbb{P}^1(\mathbb{R}) = \mathbb{R} \cup \{\infty\}$ by linear fractional transformations. Show that $G = \mathrm{SL}_2(\mathbb{Z}) \leq \mathrm{SL}_2(\mathbb{R})$ is discrete, but G does not act properly on $\mathbb{P}^1(\mathbb{R})$. (So discrete groups can act on locally compact spaces without being proper.)