Math 123 Homework Assignment #3
Friday, Math 9, 2008

Part I:

1. Suppose that P and Q are projections in $B(H)$. We say that $P \perp Q$ if $P(H) \perp Q(H)$ and that $P \leq Q$ if $P(H) \subset Q(H)$.

(a) Show that the following are equivalent.

(i) $P \perp Q$.

(ii) $PQ = QP = 0$.

(iii) $P + Q$ is a projection.

(b) Show that the following are equivalent.

(i) $P \leq Q$.

(ii) $PQ = QP = P$.

(iii) $Q - P$ is a projection.

(Hint: Note that PQP is a positive operator. Also $PQP = PQ(PQ)^*$ so that $PQP = 0$ if and only if $PQ = QP = 0$.)

2. Let $\pi : A \to B(\mathcal{H})$ be an irreducible representation of a C^*-algebra A. Suppose that $\pi(A) \cap \mathcal{K}(\mathcal{H}) \neq \{0\}$. Show that $\pi(A) \supset \mathcal{K}(\mathcal{H})$. ("If the range of an irreducible representation contains one nonzero compact operator, then it contains them all.") If you want a hint, look over the proof of Proposition AV.

3. Complete the proof of Lemma AU. That is, show that $\mathrm{CCR}(A)$ is the largest CCR ideal in A in the sense that if J is any CCR ideal in A, then $J \subset \mathrm{CCR}(A)$.

Part II:

4. Suppose that π is a non-degenerate representation of A on $\mathcal{H}$. Let $\{ e_\lambda \}$ be an approximate identity for A. Show that $\pi(e_\lambda)$ converges to I in the strong operator topology; that is, prove that $\lim_\lambda \pi(e_\lambda)\xi = \xi$ for all $\xi \in \mathcal{H}$. Conclude that $S = \{ \pi(x)\xi : x \in A, \xi \in \mathcal{H} \}$ is dense in $\mathcal{H}$. (The point is that *a priori* all we are given is that S *spans* a dense subset of $\mathcal{H}$.)

5. Let $\{ J_\alpha : 0 \leq \alpha \leq \alpha_0 \}$ be a composition series for a *separable* C^*-algebra A. Show that α_0 is countable. (Recall that α_0 is called countable if $\{ \alpha : 0 \leq \alpha < \alpha_0 \}$ is countable. Also, for each $\alpha < \alpha_0$ notice that you can find $a_\alpha \in J_{\alpha+1}$ such that $\|a_{\alpha+1} - a\| \geq 1$ for all $a \in J_\alpha$.)

6. Suppose that $\{ J_\alpha : 0 \leq \alpha \leq \alpha_0 \}$ is a composition series for a C^*-algebra A. A nondegenerate representation π if A is said to *live on the subquotient* $J_{\alpha+1}/J_\alpha$ if π is the canonical extension to A of a representation π' of $J_{\alpha+1}$ such that $\ker \pi' \supset J_\alpha$. That is, π' must be of the form $\pi' = \rho \circ q_\alpha$ where $q_\alpha : J_{\alpha+1} \to J_{\alpha+1}/J_\alpha$ is the natural map, and ρ is a nondegenerate representation of $J_{\alpha+1}/J_\alpha$. Show that every *irreducible* representation of A lives on a subquotient so that the spectrum of A can be identified with the disjoint union of the spectra of the subquotients $J_{\alpha+1}/J_\alpha$ for $\alpha < \alpha_0$.

Part III:

7. Let $\mathcal{H}$ be a separable Hilbert space with orthonormal basis $\{ e_n \}_{n=1}^\infty$. Let S be the *unilateral shift* operator $S \in B(\mathcal{H})$ defined by $S(e_n) = e_{n+1}$ for all n. Finally, let A be the unital C^*-algebra generated by S (i.e., $A = C^*(S)$), and let $\mathbb{T} = \{ z \in \mathbb{C} : |z| = 1 \}$.

(a) Show that $S^*S - SS^* = P$, where P is the rank-one projection onto $\mathbb{C}e_1$.

(b) Show that A is irreducible, and that $\mathcal{K}(\mathcal{H}) \subseteq A$.

(c) Show that if $\alpha \in \mathbb{T}$, then there is a unitary operator U in $B(\mathcal{H})$ such that $USU^* = \alpha S$.

(d) Show that the quotient $A/\mathcal{K}(\mathcal{H})$ is $*$-isomorphic to $C(\mathbb{T})$.

(e) Conclude that A is GCR, but not CCR.

(f) Describe the (equivalence classes) of irreducible representations of A.

Suggestions: In part (b), show that e_1 is cyclic for the identity representation of A. Now observe that if V is a closed invariant subspace for A, then either $e_1 \in V$ or $e_1 \in V^\perp$. In part (d), notice that the image of S in the quotient is unitary (hence normal), generates, and has spectrum $\mathbb{T}$.

8. Let $\mathcal{H}$ be a *separable* infinite dimensional Hilbert space. Recall that $T \in B(\mathcal{H})$ is said to be *below* if there is an $\epsilon > 0$ such that $|T\xi| \geq \epsilon|\xi|$ for all $\xi \in \mathcal{H}$.

(a) Show that if $T \in B(\mathcal{H})_{\text{s.a.}}$ is bounded from below, then T has a bounded inverse.

(b) If $T \in B(\mathcal{H})_{\text{s.a.}}$ and $\epsilon > 0$, then define

$$M_\epsilon = \overline{\operatorname{span}}\{\, f(T)\xi : \xi \in \mathcal{H},\ f \in C\big(\sigma(T)\big), \text{ and } f(\lambda) = 0 \text{ if } |\lambda| \leq \epsilon \,\}.$$

Show that $|T\xi| \geq \epsilon|\xi|$ for all $\xi \in M_\epsilon$, and that $TM_\epsilon = M_\epsilon$.

(c) Show that if $T \in B(\mathcal{H})_{\text{s.a.}}$ is not compact, then there is an $\epsilon > 0$ so that M_ϵ is infinite dimensional. In particular, conclude that there is a partial isometry $V \in B(\mathcal{H})$ such that V^*TV has a bounded inverse.

(d) Show that $\mathcal{K}(\mathcal{H})$ is the only non-zero proper closed ideal in $B(\mathcal{H})$.

(e) Assuming that any C^*-algebra has irreducible representations, conclude that $B(\mathcal{H})$ is not a GCR algebra.