**Math 123 *OPTIONAL* Homework Assignment on Nets**

Not not turn in.

1. Suppose that $X$ is a first countable space. Show that each $x \in X$ has a neighborhood basis of open sets $\{ U_n \}_{n=1}^{\infty}$ such that $U_{n+1} \subseteq U_n$.

2. Let $X$ and $Y$ be $1^{\text{st}}$ countable spaces.

(a) Show that $\mathcal{O} \subseteq X$ is open in $X$ if and only if every *sequence* converging to some $x \in \mathcal{O}$ is eventually in $\mathcal{O}$.

(b) Show that $F \subseteq X$ is closed if and only if every convergent *sequence* in $F$ converges to a point in $F$.

(c) Show that $f : X \to Y$ is continuous if and only if whenever $\{ x_n \}_{n=1}^{\infty}$ coverges to $x \in X$ then $\{f(x_n)\}_{n=1}^{\infty}$ converges to $f(x) \in Y$.

(d)* Let $\{ x_n \}_{n=1}^{\infty}$ be a sequence in $X$. Show that if $\{ x_n \}_{n=1}^{\infty}$ has a convergent *subnet*, then $\{ x_n \}_{n=1}^{\infty}$ has a convergent *subsequence*. (Hint: if $\{ x_n \}_{n=1}^{\infty}$ has an accumulation point, then it must have a convergent subsequence.)

(e)* Show that $1^{\text{st}}$ countability is required in part (d). (Hint: let $\ell^{\infty}$ denote the set of *bounded* sequences. If $\alpha = \{\alpha_n\}_{n=1}^{\infty} \in \ell^{\infty}$, let $I_\alpha$ be any closed bounded interval in $\mathbf{R}$ such that $\alpha_n \in I_\alpha$ for all $n$. Set

$$Z = \prod_{\alpha \in \ell^{\infty}} I_\alpha.$$

Consider the sequence $\{ x_n \}_{n=1}^{\infty}$ in the compact space $Z$ defined by $x_n(\alpha) = \alpha_n$.)

3. Let $\mathcal{H}$ be a separable infinite dimensional (complex) Hilbert space with orthonormal basis $\{ e_n \}_{n=1}^{\infty}$. Let $S = \{ \sqrt{n} e_n \}_{n=1}^{\infty}$, and let $W$ be the weak closure of $S$ in $\mathcal{H}$.

(a) Show that $0 \in W$.

(b) Observe that no *sequence* in $S$ converges weakly to 0.

(Hints: A basic weak neighborhood of 0 is of the form

$$U = \{\, h \in \mathcal{H} : |(h \mid k_j)| < \epsilon \text{ for } j = 1, \dots, n \,\}.$$

If $\sqrt{n}e_n \notin U$, then

$$\sum_{j=1}^{n} |(e_n \mid k_j)|^2 \geq \frac{\epsilon^2}{n}.$$

Observe that this can't happen for all $n$.

For the second part, observe that the principal of uniform boundedness implies that any weakly convergent *sequence* is bounded. Of course, no bounded sequence from $S$ can converge weakly to 0.)