Mathematics 11

Practice Problems on New Material for the Final

The final exam is cumulative, but will concentrate on new material not covered on the two midterm exams. These problems are only on the new material.

1. Consider the paraboloid with equation
$$z = x^2 + y^2.$$
   (a) Rewrite this equation in cylindrical coordinates and in spherical coordinates.
   (b) Let S be the portion of this paraboloid for which $z \leq 1$, oriented with the unit normal vector pointing away from the z-axis. Use the three equations for the paraboloid from part (a) to find three different parametrizations of the surface S as
$$(x, y, z) = T(u, v).$$
   In each case, identify (using one or more inequalities) the region D in the uv-plane that is mapped onto S by T, and verify that your parametrization has the correct orientation.
   (c) Use each of the three parametrizations from part (b) to express the surface area of S as an integral.

2. Suppose $\vec{F}$ is a vector field on $\mathbb{R}^3$ all of whose components have continuous first and second partial derivatives, and S is a sphere, oriented so the unit normal vector points outwards. Show that
$$\iint_S curl(\vec{F}) \cdot d\vec{S} = 0$$
in two different ways:
   (a) By using Stokes' Theorem.
   (b) By using the Divergence Theorem.

3. Use Green's Theorem to compute the area of the portion of the disc $x^2 + y^2 \leq 4$ to the right of the line $x = -1$. Verify your answer using basic geometry. (Hint: This region can be broken up into two pieces; one is the disc minus a wedge, whose area can be easily computed as a fraction of the area of the disc, and the other is a triangle.)

4. Find the flux of the vector field
$$\vec{F}(x, y, z) = \left\langle z, z, \sqrt{x^2 + y^2} \right\rangle$$
over the portion of the hyperboloid $x^2 + y^2 = z^2 + 1$ between the planes $z = 0$ and $z = \dfrac{\sqrt{3}}{3}$, oriented so the unit normal vector points away from the z-axis.
Do this directly, without using Stokes' Theorem or the Divergence Theorem.

5. Find the flux of the vector field

$$\vec{F}(x, y, z) = \langle e^y + x, 3\cos(xz) - y, z \rangle$$

through the surface S, where S is given by

$$z^2 = 4x^2 + 4y^2 \qquad 0 \leq z \leq 4,$$

oriented so the unit normal vector points downward.

6. Find the average distance from the z-axis of a point on the sphere with radius 1 and center $(0, 0, 0)$.

   (The average value of f over the surface S is defined to be the integral of f over S, divided by the surface area of S.)

7. Let P be the parallelogram with vertices at $(0, 0)$, $(1, 4)$, $(3, 1)$, and $(4, 6)$. Evaluate

$$\iint_P xy\, dA.$$

8. Evaluate

$$\int_C xy\, dx - x^2\, dy$$

where C is the circle of radius 3 centered at $(1, 0)$.

9. Compute

$$\iint_S \vec{F} \cdot d\vec{S}$$

where

$$\vec{F}(x, y, z) = x\,\vec{\imath} + y\,\vec{\jmath} + 2z\,\vec{k}$$

and S is the part of the sphere $x^2 + y^2 + z^2 = 2$ with $0 \leq z \leq 1$.

10. Find the surface area of the torus obtained by rotating the circle

$$(x - 5)^2 + y^2 = 9$$

around the y axis.

11. Consider the vector field

$$\vec{F}(x, y, z) = (3x^2yz)\vec{\imath} + (x^3z - 3x)\vec{\jmath} + (x^23y + 2z)\vec{k}.$$

   (a) Show that $\vec{F}$ is conservative.

   (b) Compute the line integral of $\vec{F}$ along the curve C parametrized by

$$\vec{r}(t) = \cos t\vec{\imath} + \sin t\vec{\jmath} + t\vec{k} \qquad 0 \leq t \leq 4\pi.$$

12. Evaluate

$$\int_C \langle e^x, e^{-x}, e^z \rangle \cdot d\vec{r}$$

where C is the boundary of the portion of the plane

$$x + y + z = 1$$

in the first octant, oriented counter-clockwise as viewed from above.

13. Identify each of the following as:

(i) undefined.
(ii) a scalar function.
(iii) a scalar function, always 0.
(iv) a vector function.
(v) a vector function, always $\vec{0}$.

given that $f : \mathbb{R}^3 \to \mathbb{R}$ and $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$.

(a) $div(grad(f))$
(b) $div(div(f))$
(c) $div(curl(f))$
(d) $grad(grad(f))$
(e) $grad(div(f))$
(f) $grad(curl(f))$
(g) $curl(grad(f))$
(h) $curl(div(f))$
(i) $curl(curl(f))$
(j) $div(grad(\vec{F}))$
(k) $div(div(\vec{F}))$
(l) $div(curl(\vec{F}))$
(m) $grad(grad(\vec{F}))$
(n) $grad(div(\vec{F}))$
(o) $grad(curl(\vec{F}))$
(p) $curl(grad(\vec{F}))$
(q) $curl(div(\vec{F}))$
(r) $curl(curl(\vec{F}))$