Mathematics 11
Practice Problems on New Material for the Final:
Sample Solutions

The final exam is cumulative, but will concentrate on new material not covered on the two midterm exams. These problems are only on the new material.

1. Consider the paraboloid with equation

$$z = x^2 + y^2.$$

(a) Rewrite this equation in cylindrical coordinates and in spherical coordinates.

Solution:

$$z = r^2$$

$$\rho \cos\varphi = \rho^2 \sin^2\varphi \qquad \rho = \frac{\cos\varphi}{\sin^2\varphi} = \cot\varphi \csc\varphi$$

(b) Let S be the portion of this paraboloid for which $z \le 1$, oriented with the unit normal vector pointing away from the z-axis. Use the three equations for the paraboloid from part (a) to find three different parametrizations of the surface S as

$$(x, y, z) = \vec{r}(u, v).$$

In each case, identify (using one or more inequalities) the region D in the uv-plane that is mapped onto S by $\vec{r}$, and verify that your parametrization has the correct orientation.

Solution:

Using rectangular coordinates, we try

$$(x, y, z) = (u, v, u^2 + v^2) \qquad u^2 + v^2 \le 1.$$

Computing

$$\vec{r}_u \times \vec{r}_v = \langle -2u,\ -2v,\ 1\rangle$$

we see we have the wrong orientation. (Here the z-component of the normal vector is positive, but if the normal vector to this paraboloid points away from the z-axis, it should point downward.) Therefore we try again, switching u and v:

$$(x, y, z) = (v, u, v^2 + u^2) \qquad v^2 + u^2 \le 1.$$

Computing

$$\vec{r}_u \times \vec{r}_v = \langle 2v,\ 2u,\ -1\rangle$$

we see the orientation is correct.

Using cylindrical coordinates, we try $u = \theta$, $v = r$, to get

$$(x, y, z) = (v \cos u,\ v \sin u,\ v^2) \qquad 0 \le u \le 2\pi \quad 0 \le v \le 1;$$

$$\vec{r}_u \times \vec{r}_v = \left\langle 2v^2 \cos u,\ 2v^2 \sin u,\ -v \right\rangle,$$

which gives the correct orientation.

Using spherical coordinates, we try $u = \theta$, $v = \varphi$, to get

$$(x, y, z) = (\cot v \cos u,\ \cot v \sin u,\ \cot^2 v,\ ) \qquad 0 \le u \le 2\pi \quad \frac{\pi}{4} \le v \le \frac{\pi}{2};$$

$$\vec{r}_u \times \vec{r}_v = (\cot v \csc^2 v) \langle 2 \cos u \cot v,\ 2 \sin u \cot v,\ -1 \rangle,$$

which gives the correct orientation.

(c) Use each of the three parametrizations from part (b) to express the surface area of S as an integral.

Solution:

$$\int_{-1}^{1} \int_{-\sqrt{1-v^2}}^{\sqrt{1-v^2}} \sqrt{4u^2 + 4v^2 + 1}\ du\ dv$$

$$\int_{0}^{1} \int_{0}^{2\pi} v\sqrt{4v^2 + 1}\, du\, dv$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_{0}^{2\pi} \cot v \csc^2 v \sqrt{4 \cot^2 v + 1}\, du\, dv$$

2. Suppose $\vec{F}$ is a vector field on $\mathbb{R}^3$ all of whose components have continuous first and second partial derivatives, and S is a sphere, oriented so the unit normal vector points outwards. Show that

$$\iint_S curl(\vec{F}) \cdot d\vec{S} = 0$$

in two different ways:

(a) By using Stokes' Theorem.

Solution: Use a horizontal plane through the center of the sphere to divide S into two hemispheres, meeting at a circle C. Let γ denote C oriented counterclockwise as seen from above, and $-\gamma$ denote C with the opposite orientation.

Since the top hemisphere S_1 is oriented with unit normal vector pointing upward, its boundary is γ; since the bottom hemisphere S_2 is oriented with unit normal vector pointing downward, its boundary is $-\gamma$. By Stokes' Theorem,

$$\iint_S curl(\vec{F}) \cdot d\vec{S} = \iint_{S_1} curl(\vec{F}) \cdot d\vec{S} + \iint_{S_2} curl(\vec{F}) \cdot d\vec{S} = \int_\gamma \vec{F} \cdot d\vec{r} + \int_{-\gamma} \vec{F} \cdot d\vec{r}.$$

Since $-\gamma$ is γ with the opposite orienation, we have

$$\int_{-\gamma} \vec{F} \cdot d\vec{r} = -\int_{\gamma} \vec{F} \cdot d\vec{r},$$

and so

$$\iint_S curl(\vec{F}) \cdot d\vec{S} = \int_{\gamma} \vec{F} \cdot d\vec{r} + \int_{-\gamma} \vec{F} \cdot d\vec{r} = \int_{\gamma} \vec{F} \cdot d\vec{r} + \left(-\int_{\gamma} \vec{F} \cdot d\vec{r} \right) = 0.$$

(b) By using the Divergence Theorem.

Solution: We know (from Clairaut's Theorem) that $div(curl(\vec{F})) = 0$. By the Divergence Theorem, letting E denote the solid ball with boundary S, we have

$$\iint_S curl(\vec{F}) \cdot d\vec{S} = \iiint_E (div(curl(\vec{F}))\, dV = \iiint_E 0\, dV = 0$$

3. Use Green's Theorem to compute the area of the portion of the disc $x^2 + y^2 \leq 4$ to the right of the line $x = -1$. Verify your answer using basic geometry. (Hint: This region can be broken up into two pieces; one is the disc minus a wedge, whose area can be easily computed as a fraction of the area of the disc, and the other is a triangle.)

Solution:

Let D denote the region in question, bounded by a portion of the circle of radius 2 around the origin and a portion of the line $x = -1$. The circle and the line intersect at the points $(-1, -\sqrt{3})$ $(\theta = -\frac{2\pi}{3})$ and $(-1, \sqrt{3})$ $(\theta = \frac{2\pi}{3})$. Let C denote the portion of the circle that is part of the boundary of D, and L the portion of the line that is the boundary of D. By Green's Theorem, we have

$$area(D) = \iint_D 1\, dA = \iint \frac{\partial}{\partial x}\left(\frac{x}{2}\right) - \frac{\partial}{\partial y}\left(\frac{-y}{2}\right) dA = \int_{\partial D} \frac{-y}{2}\, dx + \frac{x}{2}\, dy =$$

$$\int_C \frac{-y}{2}\, dx + \frac{x}{2}\, dy + \int_L \frac{-y}{2}\, dx + \frac{x}{2}\, dy.$$

To have the positive orientation, C must be oriented counterclockwise, and L from top to bottom. We can parametrize C by $\langle x, y \rangle = \langle 2\cos t,\, 2\sin t \rangle$, $-\dfrac{2\pi}{3} \leq t \leq \dfrac{2\pi}{3}$, and get

$$\int_C \frac{-y}{2}\, dx + \frac{x}{2}\, dy = \int_{-\frac{2\pi}{3}}^{\frac{2\pi}{3}} \langle -\sin t, \cos t \rangle \cdot \langle -2\sin t,\, 2\cos t \rangle\, dt = \frac{8\pi}{3}.$$

On L, $x = -1$ and $-\sqrt{3} \le y \le \sqrt{3}$, but the orientation is from $y = \sqrt{3}$ to $y = -\sqrt{3}$, so

$$\int_L \frac{-y}{2}\,dx + \frac{x}{2}\,dy = \int_{\sqrt{3}}^{-\sqrt{3}} -\frac{1}{2}\,dy = \sqrt{3}.$$

Thus the area of D is

$$\int_C \frac{-y}{2}\,dx + \frac{x}{2}\,dy + \int_L \frac{-y}{2}\,dx + \frac{x}{2}\,dy = \frac{8\pi}{3} + \sqrt{3}.$$

To check this using basic geometry: The entire disc of radius 2 has area 4π, and the disc minus the wedge (that is, the portion of the disc given by $-\frac{2\pi}{3} \le \theta \le \frac{2\pi}{3}$) is two-thirds of the entire disc, so its area is $\frac{8\pi}{3}$. To this we must add the area of the triangle with corners, $(0,0)$, $(-1,\sqrt{3})$, and $(-1,-\sqrt{3})$, which is $\sqrt{3}$. Hence the area of D is $\frac{8\pi}{3} + \sqrt{3}$.

4. Find the flux of the vector field

$$\vec{F}(x,y,z) = \left\langle z, z, \sqrt{x^2+y^2} \right\rangle$$

over the portion of the hyperboloid $x^2 + y^2 = z^2 + 1$ between the planes $z = 0$ and $z = \frac{\sqrt{3}}{3}$, oriented so the unit normal vector points away from the z-axis.
Do this directly, without using Stokes' Theorem or the Divergence Theorem.

Solution: The equation of the hyperboloid can be expressed in cylindrical coordinates as $r^2 = 1 + z^2$, or $z = \sqrt{r^2-1}$; we have the portion $0 \le z \le \frac{\sqrt{3}}{3}$, or $1 \le r \le \frac{2\sqrt{3}}{3}$. Parametrizing the surface using $u = r$ and $v = \theta$ we get

$$\langle x,y,z\rangle = \vec{r} = \left\langle u\cos v,\ u\sin v,\ \sqrt{u^2-1} \right\rangle$$

$$1 \le u \le \frac{2\sqrt{3}}{3} \qquad 0 \le v \le 2\pi$$

$$\vec{r}_u \times \vec{r}_v = \left\langle -\frac{u^2\cos v}{\sqrt{u^2-1}},\ -\frac{u^2\sin v}{\sqrt{u^2-1}},\ u \right\rangle.$$

This has the wrong orientation, so we put a minus sign in front of the integral to account for that:

$$\iint_S \vec{F}\cdot d\vec{S} = -\int_0^{2\pi}\int_1^{\frac{2\sqrt{3}}{3}} \left\langle \sqrt{u^2-1},\ \sqrt{u^2-1},\ u \right\rangle \cdot \left\langle -\frac{u^2\cos v}{\sqrt{u^2-1}},\ -\frac{u^2\sin v}{\sqrt{u^2-1}},\ u \right\rangle du\,dv =$$

$$-\int_0^{2\pi}\int_1^{\frac{2\sqrt{3}}{3}} u^2(\cos v+\sin v+1)\,du\,dv = -\frac{2\pi(8\sqrt{3}-9)}{27}.$$

Note: This can also be done using the Divergence Theorem, as in the solution to the next problem: by adding two discs to S we can form a closed surface.

5. Find the flux of the vector field

$$\vec{F}(x,y,z) = \langle e^y + x, 3\cos(xz) - y, z\rangle$$

through the surface S, where S is given by

$$z^2 = 4x^2 + 4y^2 \qquad 0 \le z \le 4,$$

oriented so the unit normal vector points downward.

Solution: We could do this directly, but we can also use the Divergence Theorem. S is the portion of the cone $z^2 = 4(x^2+y^2)$ with $0 \le z \le 4$, oriented with $\vec{N}$ pointing outward; if we let D be the disc $x^2+y^2 \le 4$, $z = 4$, oriented with $\vec{N}$ pointing upward, then $S \cup D$ is the boundary of the solid region E given by $2\sqrt{x^2+y^2} \le z \le 4$, a right circular cone of height 4 and base radius 2.

The divergence of the vector field $\vec{F}$ is 1, so applying the Divergence Theorem, we get

$$\iint_S \vec{F}\cdot d\vec{S} + \iint_D \vec{F}\cdot d\vec{S} = \iiint_E div(\vec{F})\,dV = \iiint_E 1\,dV = volume(E) = \frac{16\pi}{3}.$$

On D, we have $z = 4$ and $\vec{N} = \langle 0,0,1\rangle$, so

$$\iint_D \vec{F}\cdot d\vec{S} = \iint_D \vec{F}\cdot\vec{N}\,dS = \iint_D z\,dS = \iint_D 4\,dS = 4(area(D)) = 16\pi.$$

Solving,

$$\iint_S \vec{F}\cdot d\vec{S} = -\frac{32\pi}{3}.$$

6. Find the average distance from the z-axis of a point on the sphere with radius 1 and center $(0,0,0)$.

(The average value of f over the surface S is defined to be the integral of f over S, divided by the surface area of S.)

Solution: The distance from the z-axis of the point (x,y,z) is $\sqrt{x^2+y^2}$, so we need to compute $\iint_S \sqrt{x^2+y^2}\,dS$, and divide by the surface area of S.

The sphere S can be parametrized using spherical coordinates, setting $u = \varphi$ and $v = \theta$, by

$$\langle x,y,z\rangle = \vec{r} = \langle \sin u\cos v, \sin u\sin v, \cos u\rangle$$

$$0 \le u \le \pi \qquad 0 \le v \le 2\pi$$

$$dS = |\vec{r}_u \times \vec{r}_v| \; du\, dv = \left|\left\langle \sin^2 u \cos v,\, \sin^2 u \sin v,\, \sin u \cos u \right\rangle\right| \; du\, dv = \sin u\, du\, dv.$$

$$area(S) = \iint_S dS = \int_0^{2\pi} \int_0^{\pi} \sin u\, du\, dv = 4\pi$$

$$\iint_S \sqrt{x^2 + y^2}\, dS = \int_0^{2\pi} \int_0^{\pi} (\sin u)\, \sin u\, du\, dv = \pi^2$$

Hence the average distance from the z axis of a point on the sphere is

$$\frac{\pi^2}{4\pi} = \frac{\pi}{4}.$$

7. Let P be the parallelogram with vertices at $(0,0)$, $(1,4)$, $(3,2)$, and $(4,6)$. Evaluate

$$\iint_P xy\, dA.$$

Solution: P is the image of the square S in the uv-plane with corners $(0,0)$, $(0,1)$, $(1,0)$, and $(1,1)$ under the linear transformation

$$(x,y) = T(u,v) = \langle 3u + v,\, 2u + 4v \rangle \qquad \frac{\partial(x,y)}{\partial(u,v)} = \left| \det \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix} \right| = 10.$$

Using this change of variables,

$$\iint_P xy\, dx\, dy = \iint_S (3u+v)(2u+4v) 10\, du\, dv = \int_0^1 \int_0^1 60u^2 + 140uv + 40v^2\, du\, dv = \frac{205}{3}.$$

8. Evaluate

$$\int_C xy\, dx - x^2\, dy$$

where C is the circle of radius 3 centered at $(1,0)$.

Solution: We can do this directly, or by using Green's Theorem. Since C is a closed curve, we can assume it has the positive (counterclockwise) orientation.

To compute the integral directly, parametrize C by $\langle x, y \rangle = \langle \cos t, \sin t \rangle$, $0 \le t \le 2\pi$, so $dx = -\sin t\, dt$ and $dy = \cos t\, dt$. Then

$$\int_C xy\, dx - x^2\, dy = \int_0^{2\pi} (\cos t \sin t)(-\sin t) + (-\cos^2 t)(\cos t)\, dt = \int_0^{2\pi} -\cos t\, dt = 0.$$

To use Green's Theorem, let D be the unit disc, so $\partial D = C$. By Green's Theorem

$$\int_C xy\,dx - x^2\,dy = \iint_D \frac{\partial}{\partial x}(-x^2) - \frac{\partial}{\partial y}(xy)\,dA = \iint_D -3x\,dA = 0.$$

(This last integral can be seen to be zero by symmetry: $-3x$ is an odd function of x, and D is symmetric about $x = 0$.)

9. Compute

$$\iint_S \vec{F} \cdot d\vec{S}$$

where

$$\vec{F}(x, y, z) = x\,\vec{i} + y\,\vec{j} + 2z\,\vec{k}$$

and S is the part of the sphere $x^2 + y^2 + z^2 = 2$ with $0 \le z \le 1$.

Solution: We aren't given an orientation for S, but since S is part of a sphere it makes sense to take $\vec{N}$ pointing outwards, or (since we're on the top half of the sphere) upwards.

Parametrize S using spherical coordinates, with $\rho = \sqrt{2}$, $u = \varphi$, $v = \theta$:

$$\vec{r} = \langle x, y, z\rangle = \left\langle \sqrt{2}\sin u \cos v,\ \sqrt{2}\sin u \sin v,\ \sqrt{2}\cos u \right\rangle$$

$$\frac{\pi}{4} \le u \le \frac{\pi}{2} \qquad 0 \le v \le 2\pi$$

$$\vec{r}_u \times \vec{r}_v = \left\langle 2\sin^2 u \cos v,\ 2\sin^2 u \sin v,\ 2\sin u \cos u \right\rangle,$$

which has the correct orientation (as the z-component is a positive multiple of z).

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \langle x, y, 2z\rangle \cdot d\vec{S} =$$

$$\int_0^{2\pi} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left\langle \sqrt{2}\sin u \cos v,\ \sqrt{2}\sin u \sin v,\ 2\sqrt{2}\cos u \right\rangle \cdot$$

$$\left\langle 2\sin^2 u \cos v,\ 2\sin^2 u \sin v,\ 2\sin u \cos u \right\rangle du\,dv =$$

$$\int_0^{2\pi} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 2\sqrt{2}(\sin u + \sin u \cos^2 u)\,du\,dv = \frac{14\pi}{3}.$$

10. Find the surface area of the torus obtained by rotating the circle

$$(x - 5)^2 + y^2 = 9$$

around the y axis.

Solution: We need to parametrize the torus. First, we can parametrize the circle as

$$x - 5 = 3\cos t \qquad y = 3\sin t$$

$$x = 3\cos t + 5 \qquad y = 3\sin t \qquad 0 \le t \le 2\pi$$

Now if we rotate a point $(x, y) = (a, b) = (3\cos t + 5, 3\sin t)$ on this circle about the y-axis through an angle θ, the y-coordinate remains b, and the distance from the y-axis remains a, but (thinking polar coordinates in the xz-plane), instead of $(x, z) = (a, 0)$, we have $(x, z) = (a\cos\theta, a\sin\theta)$. That is, the coordinates of the rotated point are

$$(x, y, z) = (a\cos\theta, b, a\sin\theta) = ((3\cos t + 5)\cos\theta, 3\sin t, (3\cos t + 5)\sin\theta).$$

We use $u = t$ and $v = \theta$ to parametrize S:

$$\vec{r} = \langle x, y, z\rangle = \langle (3\cos u + 5)\cos v,\ 3\sin u,\ (3\cos u + 5)\sin v\rangle$$

$$0 \le u \le 2\pi \qquad 0 \le v \le 2\pi$$

$$\vec{r}_u \times \vec{r}_v = \langle 3\cos v\cos u(3\cos u + 5),\ 3\sin u(3\cos u + 5),\ 3\sin v\cos u(3\cos u + 5)\rangle\,.$$

$$area(S) = \iint_S dS = \int_0^{2\pi}\int_0^{2\pi} |\vec{r}_u \times \vec{r}_v|\,du\,dv = \int_0^{2\pi}\int_0^{2\pi} 9\cos u + 15\,du\,dv = 60\pi^2.$$

11. Consider the vector field

$$\vec{F}(x, y, z) = (3x^2yz)\vec{i} + (x^3z - 3x)\vec{j} + (x^23y + 2z)\vec{k}.$$

(a) Show that $\vec{F}$ is conservative.

Solution: Check that the curl of $\vec{F}$ equals $\vec{0}$ everywhere.

Actually, it doesn't, because there was a typo in this problem. The field should have been

$$\vec{F}(x, y, z) = (3x^2yz)\vec{i} + (x^3z - 3y)\vec{j} + (x^3y + 2z)\vec{k}.$$

(b) Compute the line integral of $\vec{F}$ along the curve C parametrized by

$$\vec{r}(t) = \cos t\vec{i} + \sin t\vec{j} + t\vec{k} \qquad 0 \le t \le 4\pi.$$

Solution: If we have the correct field, which is actually conservative, there are three ways to do the problem:

(1.) Directly: Use the parametrization of the curve to compute the line integral. This is the hard way.

(2.) Using path independence to convert the integral into an integral along an easier path with the same endpoints, say the line segment $x = 1$, $y = 0$, $0 \le z \le 4\pi$. This gives us

$$\int_C 3x^2yz\,dx + (x^2z - 3y)\,dy + (x^3y + 2z)\,dz = \int_0^{4\pi} 2z\,dz = 16\pi^2.$$

(3.) Finding a potential function for $\vec{F}$ (it's $f(x, y, z) = x^2yz - \frac{3y^2}{2} + 2z$) and evaluating it at the endpoints:

$$\int_C \vec{F}\,d\vec{r} = f(1, 0, 4\pi) - f(1, 0, 0) = 16\pi^2 - 0 = 16\pi^2.$$

12. Evaluate

$$\int_C \left\langle e^x, e^{-x}, e^z \right\rangle \cdot d\vec{r}$$

where C is the boundary of the portion of the plane

$$x + y + z = 1$$

in the first octant, oriented counter-clockwise as viewed from above.

Solution: The easy way to do this is by using Stokes' Theorem. Let S denote the surface of which C is the boundary, and $vecF = \langle e^x, e^{-x}, e^z \rangle$. Then

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S (curl\vec{F}) \cdot d\vec{S} = \iint_S \left\langle 0, 0, -e^{-x} \right\rangle \cdot d\vec{S}.$$

If we parametrize S by setting $x = u$ and $y = v$, we have

$$\vec{r} = \langle x, y, z \rangle = \langle u, v, 1 - u - v \rangle$$

$$0 \le u \le 1 \qquad 0 \le v \le 1 - u$$

$$\vec{r}_u \times \vec{r}_v = \langle 1, 1, 1 \rangle\,.$$

We can see this has the correct orientation.

$$\iint_S \left\langle 0, 0, -e^{-x} \right\rangle \cdot d\vec{S} = \int_0^1 \int_0^{1-u} -e^{-u}\,dv\,du = \int_0^1 ue^{-u} - e^{-u}\,du.$$

This integral is not intractable (use integration by parts), but it becomes even easier if we reverse the order of integration:

$$\int_0^1 \int_0^{1-v} -e^{-u}\,du\,dv = \int_0^1 e^{v-1} - 1\,dv = (e^{v-1} - v)\Big|_{v=0}^{v=1} = -e^{-1}.$$

13. Identify each of the following as:

   (i) undefined.
   (ii) a scalar function.
   (iii) a scalar function, always 0.
   (iv) a vector function.
   (v) a vector function, always $\vec{0}$.

   given that $f : \mathbb{R}^3 \to \mathbb{R}$ and $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$.

   (a) $div(grad(f))$
   Solution: scalar (not 0, e.g. if $f = y^2$).
   (b) $div(div(f))$
   Solution: undefined.
   (c) $div(curl(f))$
   Solution: undefined.
   (d) $grad(grad(f))$
   Solution: undefined.
   (e) $grad(div(f))$
   Solution: undefined.
   (f) $grad(curl(f))$
   Solution: undefined.
   (g) $curl(grad(f))$
   Solution: vector, $\vec{0}$.
   (h) $curl(div(f))$
   Solution: undefined.
   (i) $curl(curl(f))$
   Solution: undefined.
   (j) $div(grad(\vec{F}))$
   Solution: undefined.
   (k) $div(div(\vec{F}))$
   Solution: undefined.
   (l) $div(curl(\vec{F}))$
   Solution: vector, $\vec{0}$.
   (m) $grad(grad(\vec{F}))$
   Solution: undefined.

(n) $grad(div(\vec{F}))$

Solution: vector (not $\vec{0}$, e.g. if $\vec{F} = \langle y^2, y^2, 0 \rangle$).

(o) $grad(curl(\vec{F}))$

Solution: undefined.

(p) $curl(grad(\vec{F}))$

Solution: undefined.

(q) $curl(div(\vec{F}))$

Solution: undefined.

(r) $curl(curl(\vec{F}))$

Solution: vector (not $\vec{0}$, e.g. if $\vec{F} = \langle y^2, y^2, 0 \rangle$).