Mathematics 11
Practice Exam 2
Sample Solutions

1. Consider the function

$$f(x, y) = 2xy + x^2.$$

(a) Find all critical points of f.

Solution: $\nabla f = \langle f_x,\, f_y\rangle = \langle 2y + 2x,\, 2x\rangle$, so the only critical point of the function is $(0, 0)$.

(b) For each critical point, determine whether it is a local maximum, local minimum, or saddle point.

Solution: Using the second derivative test, $D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 2 & 2 \\ 2 & 0 \end{vmatrix} = -4 < 0$, so $(0, 0)$ is a saddle point.

(c) Consider the rectangular region

$$D = \{(x, y) \mid -1 \leq x \leq 1,\ -1 \leq y \leq 1\}.$$

Determine the absolute <u>maximum</u> value of f on D, and state the point(s) where f attains this value.

Solution: We must check critical points inside D and boundary points of D. At the critical point $(0, 0)$ we have $f(0, 0) = 0$. On the boundary, we can reason without using calculus: The maximum value of x^2 is where $|x|$ is largest, along the left and right edges of D, where $x^2 = 1$. The maximum value of xy occurs where x and y have the same sign and are each as large as possible, at the upper right and lower left corners of$D$, where $xy = 1$. Hence the maximum value of $x^2 + xy$ occurs at the upper right and lower left corners, where $f(x, y) = x^2 + 2xy = 3$. Since $3 > 0$, this is the maximum value of $f(x, y)$ on D.

2. Let

$$F(x, y) = \left\langle ye^{xy}\sin y + \cos y,\ xe^{xy}\sin y + e^{xy}\cos y\right\rangle.$$

Is F conservative?

Solution: Letting $F = \langle P,\, Q\rangle$, we see

$$\frac{\partial P}{\partial y} = e^{xy}\sin y + xye^{xy}\sin y + ye^{xy}\cos y - \sin y \qquad \frac{\partial Q}{\partial x} = e^{xy}\sin y + xye^{xy}\sin y + ye^{xy}\cos y.$$

Since these partial derivatives are not equal, F is not conservative

If so, find a potential function for F.

Solution: F is not conservative.

If not, find a closed curve (a loop) C for which $\displaystyle\int_C F \cdot d\vec{r} \neq 0$, and determine the value of $\displaystyle\int_C F \cdot d\vec{r}$.

Solution: Notice that F is very close to being conservative; it would be if not for the term $\cos y$ in the first coordinate. If we write $F = G + H$, where $G(x, y) = \langle ye^{xy} \sin y,\, xe^{xy} \sin y + e^{xy} \cos y\rangle$ and $H(x, y) = \langle \cos y,\, 0\rangle$, then G is conservative, and

$$\int_C F \cdot d\vec{r} = \int_C G \cdot d\vec{r} + \int_C H \cdot d\vec{r} = 0 + \int_C H \cdot d\vec{r},$$

so we need to find a closed curve C on which $\int_C H \cdot d\vec{r} = \int_C \cos y\, dx \neq 0$.

One such curve C is the boundary of the rectangle with corners $(0, 0)$, $(1, 0)$, $(1, \frac{\pi}{2})$, $(0, \frac{\pi}{2})$. The integral of $\cos y$ with respect to x along the vertical edges of the rectangle is 0 because $dx = 0$, and along the top edge it is 0 because $\cos y = 0$. Along the bottom edge we get $\int_0^1 \cos 0\, dx = 1$. Therefore $\int_C H \cdot d\vec{r} = 1$.

3. Let R be the rectangle with vertices at $(1, 2)$, $(6, 2)$, $(6, 5)$, $(1, 5)$, and let C be the curve that traverses the sides of R counterclockwise.

   Suppose $f(x, y)$ is a function on $\mathbb{R}^2$ satisfying $3 \leq f(x, y) \leq 7$ for all x, y. What is the maximum possible value of

$$\int_C f(x, y)dx + f(x, y)dy?$$

   Solution: First, let us consider the bottom edge of the rectangle. Along this edge we have $\int_C f(x, y)dx + f(x, y)dy = \int_1^6 f\, dx$, which must be between $\int_1^6 3\, dx$ and $\int_1^6 7\, dx$, or between 15 and 35; the maximum possible value is 35.

   The top edge is oriented in the reverse direction, so we have $\int_C f(x, y)dx + f(x, y)dy = \int_6^1 f\, dx = -\int_1^6 f\, dx$, which must be between -15 and -35; the maximum possible value is -15.

   Similar reasoning applied to the sides of the rectangle, which have length 3, tells us that along the right side the maximum value of the integral is 21 and along the left side the maximum value is -9.

   In total, the maximum possible value of the integral is $35 - 15 + 21 - 9 = 32$.

4. Find

$$\iiint_E x\, dV$$

   where E is the region in $\mathbb{R}^3$ above the xy-plane, below the surface $z = 1 - x^2$, and between the planes $y = 0$ and $y = 4$.

Solution: The surface $z = 1 - x^2$ forms a tunnel with parabolic cross-section, intersecting the xy-plane in the lines $x = \pm 1$.

$$\int_{-1}^{1}\int_{0}^{4}\int_{0}^{1-x^2} x\,dz\,dy\,dx = \int_{-1}^{1}\int_{0}^{4} x - x^3\,dy\,dx = \int_{-1}^{1} 4x - 4x^3\,dx = 0.$$

We could have figured this out in advance from the symmetry of the region across the plane $x = 0$ and the fact that the integrand is an odd function of x.

5. Do NOT evaluate the following integrals.

   (a) Rewrite
   $$\int_0^{\pi}\int_0^4\int_{-r^2}^{0} zr^4\cos\theta\; dz\; dr\; d\theta$$
   as an integral or sum of integrals in rectangular coordinates.

   Solution: $\displaystyle\int_{-4}^{4}\int_{0}^{\sqrt{16-x^2}}\int_{-(x^2+y^2)}^{0} xz(x^2+y^2)\,dz\,dy\,dx.$

   (b) Rewrite
   $$\int_0^1\int_{1-y}^{1} x^2 + y^2\,dx\,dy$$
   as an integral or sum of integrals in polar coordinates.

   Solution: $\displaystyle\int_0^{\frac{\pi}{4}}\int_{\frac{1}{\sin\theta+\cos\theta}}^{\frac{1}{\cos\theta}} r^3\,dr\,d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_{\frac{1}{\sin\theta+\cos\theta}}^{\frac{1}{\sin\theta}} r^3\,dr\,d\theta.$

6. The lemniscate of Bernoulli is a curve in the plane defined by the equation
$$(x^2+y^2)^2 = x^2 - y^2$$
(see the picture). Let R be the portion of the right petal of the lemniscate above the x-axis. Evaluate $\displaystyle\iint_R xy\,dA$.

Solution: In polar coordinates, the equation of the curve is $r^4 = r^2(\cos^2\theta - \sin^2\theta)$, or $r^2 = \cos 2\theta$. Setting this up as an integral in polar coordinates, we have

$$\int_0^{\frac{\pi}{2}}\int_0^{\sqrt{\cos 2\theta}} (r^2\sin\theta\cos\theta) r\,dr\,d\theta = \int_0^{\frac{\pi}{2}} \frac{1}{4}(\cos^2 2\theta\sin\theta\cos\theta)\,d\theta =$$

$$\int_0^{\frac{\pi}{2}} \frac{1}{8}(\cos^2 2\theta\sin 2\theta)\,d\theta = -\frac{1}{48}\cos^3 2\theta\Big|_{\theta=0}^{\theta=\frac{\pi}{2}} = \frac{1}{24}.$$

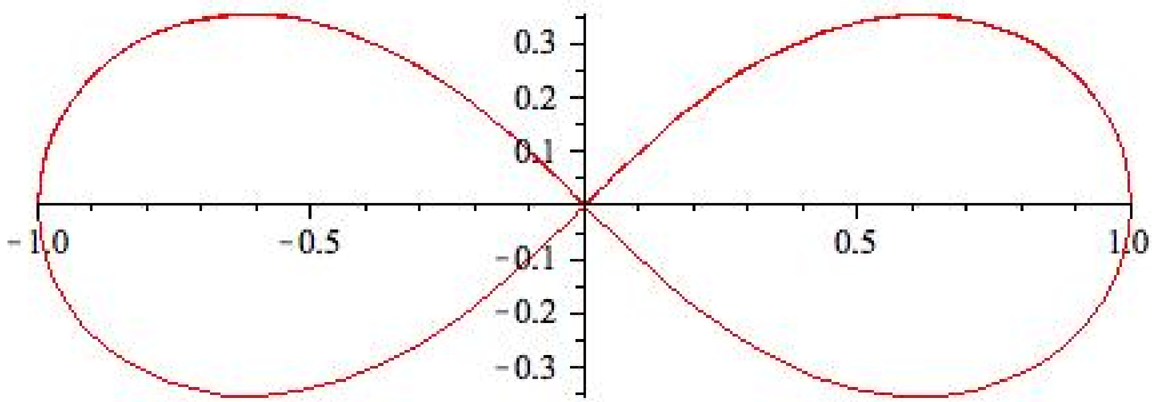

Figure 1: The lemniscate of Bernoulli (problem 6).