Math 11
Fall 2012

Practice Exam I

This practice exam is intended to give you an idea of the kinds of problems we consider putting on an exam, and of the possible length of a midterm exam. Any topic that appeared on a homework problem may appear on the exam, as may anything covered in class or in the reading. You will notice that some of the following problems are not quite like any homework problem, but require you to think about what you have learned and apply it in new ways. Others are quite straightforward, and very similar to homework problems.

Here is a good way to use the practice exam to study. *First*, before you even look at the practice exam, study. *Then*, when you feel prepared to take the exam, take the practice exam in a single sitting in a quiet place. If you do well on these sample problems under those conditions, you are probably ready for the exam; if not, you will find some area or areas you should review more thoroughly.

Following the actual practice exam are some additional practice problems of the sort that could appear on an exam.

Disclaimer: A single exam, or practice exam, or review sheet, cannot possibly include every specific topic or problem that might be on an exam. There may be things on the actual exam that were covered in class, reading, or homework, but do not appear here.

1. (a) Find a equation for the plane containing the points $(-1, 0, 3)$, $(0, 1, 2)$, and $(1, 1, -1)$.

Solution: Two vectors parallel to the plane:

$$\langle 1, 1, -1\rangle - \langle -1, 0, 3\rangle = \langle 2, 1, -4\rangle$$

$$\langle 1, 1, -1\rangle - \langle 0, 1, 2\rangle = \langle 1, 0, -3\rangle$$

A normal vector to the plane:

$$\langle 2, 1, -4\rangle \times \langle 1, 0, -3\rangle = \langle -3, 2, -1\rangle$$

An equation for the plane (using this normal vector and the point $(1, 1, -1)$):

$$-3(x-1) + 2(y-1) - (z+1) = 0,$$

or

$$3x + 2y - z = 0.$$

(b) Find an equation for the plane parallel to the plane in part (a) and containing the point $(1, 1, 1)$.

Solution:
This plane has the same normal vector $\langle -3, 2, -1\rangle$ as the first.

$$-3(x-1) + 2(y-1) - (z-1) = 0,$$

or

$$-3x + 2y - z = -2$$

2. (a) Show that the curve $\left\langle 2\sqrt{5}\cos\theta,\ 2\sqrt{5}\sin\theta,\ 4\right\rangle$ is the intersection of the sphere of radius 6 centered at the origin and the paraboloid $5z = x^2 + y^2$.

Solution: The equation of the sphere is $x^2 + y^2 + z^2 = 36$, or

$$x^2 + y^2 = 36 - z^2,$$

so the intersection is given by

$$5z = x^2 + y^2 = 36 - z^2.$$

The first and third terms give $5z = 36 - z^2$, or $z^2 + 5z - 36 = 0$, or $(z+9)(z-4) = 0$, or $z = -9, 4$. Since $5z = x^2 + y^2$ implies $z \geq 0$, the solution $z = -9$ does not apply, and the intersection lies in the plane $z = 4$. Therefore a function parametrizing it has the form

$$\vec{r} = \langle x, y, 4\rangle,$$

which this function does.

Now we find the functions x and y. As $z = 4$, the equation of the paraboloid gives $x^2 + y^2 = 20$, which is a circle of radius $\sqrt{20} = 2\sqrt{5}$. We can parametrize this circle in the xy-plane by $x = 2\sqrt{5}\cos\theta$, $y = 2\sqrt{5}\sin\theta$. Therefore our curve in three dimensions can be parametrized by

$$\vec{r} = \left\langle 2\sqrt{5}\cos\theta,\ 2\sqrt{5}\sin\theta,\ 4 \right\rangle.$$

Alternative Solution: We can see that the curve $\langle 2\sqrt{5}\cos\theta,\ 2\sqrt{5}\sin\theta,\ 4\rangle$ is a circle $x = r\cos\theta$, $y = r\sin\theta$, in the plane $z = 4$. By graphing the sphere and the paraboloid, we can see that if they intersect, it is in a circle. Therefore, to check that this is the correct circle, we can plug the values $x = 2\sqrt{5}\cos\theta$, $y = 2\sqrt{5}\sin\theta$, $z = 4$ into both equations and see that both equations hold.

(b) Find the arclength of the curve $\vec{r}(t) = \langle 7\sqrt{2}t,\ e^{7t},\ e^{-7t}\rangle$, $0 \le t \le 1$.

Solution:

$$\frac{d\vec{r}}{dt} = \left\langle 7\sqrt{2},\ 7e^{7t},\ -7e^{-7t} \right\rangle = 7\left\langle \sqrt{2},\ e^{7t},\ -e^{-7t} \right\rangle$$

$$\frac{ds}{dt} = 7\sqrt{2 + e^{14t} + e^{-14t}} = 7\sqrt{(e^{7t} + e^{-7t})^2} = 7e^{7t} + 7e^{-7t}$$

$$\text{arclength} = \int_0^1 7e^{7t} + 7e^{-7t}\,dt = (e^{7t} - e^{-7t})\Big|_0^1 = (e^7 - e^{-7}) - (e^0 - e^0) = (e^7 - e^{-7})$$

3. Find the limits or show they do not exist.

(a)

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + 2xy + y^2}$$

Solution: The limit does not exist. Along the line $x = 0$ the function has value $\dfrac{-y^2}{y^2} = -1$, and along the line $y = 0$ the function has value $\dfrac{x^2}{x^2} = 1$. Therefore as $(x, y) \to (0, 0)$ along these two different lines, the value of the function approaches -1 in one case and 1 in the other, showing the limit does not exist.

(b)

$$\lim_{(x,y)\to(0,0)} \frac{x^4 + x^2y^2 - y^4}{x^2 + y^2}$$

Solution: The limit is 0. We can express the function as a sum and look at each limit independently.

Since $x^2 + y^2 \geq x^2$, we have

$$0 \leq \frac{x^4}{x^2 + y^2} \leq \frac{x^4}{x^2} = x^2.$$

Since as $(x, y) \to (0, 0)$ we know $x \to 0$, by the Squeeze Theorem we have

$$\lim_{(x,y)\to(0,0)} \frac{x^4}{x^2 + y^2} = 0.$$

Exactly the same argument works to show

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y^2}{x^2 + y^2} = 0,$$

and, using $x^2 + y^2 \geq y^2$,

$$\lim_{(x,y)\to(0,0)} \frac{-y^4}{x^2 + y^2} = 0.$$

(c)

$$\lim_{(x,y)\to(1,2)} \frac{\cos(4x^2 + y^2 - 8) - 1}{4x^2 + y^2 - 8}$$

Solution: The limit is 0.
Setting $t = 4x^2 + y^2 - 8$, we have

$$\lim_{(x,y)\to(1,2)} t = 0,$$

so (using l'Hôpital's Rule at a key point)

$$\lim_{(x,y)\to(1,2)} \frac{\cos(4x^2 + y^2 - 8) - 1}{4x^2 + y^2 - 8} = \lim_{t\to 0} \frac{\cos(t) - 1}{t} = \lim_{t\to 0} \frac{-\sin(t)}{1} = 0$$

4. Let S be the surface $x + y = 2z^2$, and γ be the intersection of S with the plane $x = y$.

(a) Find a function f such that S is a level surface of f, and find a vector $\vec{n}$ perpendicular to S at the point $(4, 4, 2)$.

Solution: S is a level surface of $f(x, y, z) = x + y - 2z^2$, and $\nabla f(x, y, z) = \langle 1, 1, -4z \rangle$ is normal to S at (x, y, z).

$$\vec{n} = \langle 1, 1, -8 \rangle .$$

(b) Find a function $\vec{r}$ parametrizing the curve γ, and show directly (without using the fact that $\vec{n}$ is perpendicular to S) that at the point $(4, 4, 2)$, the vector $\vec{n}$ is perpendicular to the direction of γ. (Hint: Let $z = t$.)

Solution: $\vec{r}(t) = \langle t^2, t^2, t\rangle$. At $(4, 4, 2)$, we have $t = 2$, so $\vec{r}\,'(t) = \langle 2t, 2t, 1\rangle = \langle 4, 4, 1\rangle$. The derivative of $\vec{r}$ is tangent to γ and points in the direction of γ so to show $\vec{n}$ is perpendicular to the direction of γ at this point, we check

$$\vec{r}\,'(2) \cdot \vec{n} = \langle 4, 4, 1\rangle \cdot \langle 1, 1, -8\rangle = 0.$$

(c) Use the chain rule to compute $\dfrac{d}{dt} f(\vec{r}(t))$.

Solution:

$$\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t) = \langle 1, 1, -4z\rangle \cdot \langle 2t, 2t, 1\rangle = \langle 1, 1, -4t\rangle \cdot \langle 2t, 2t, 1\rangle = 0.$$

5. Use partial derivatives to approximate $\sqrt{81.2} - \sqrt[3]{124.8}$.

Solution: Use the tangent approximation $L(x, y)$ to $f(x, y) = \sqrt{x} - \sqrt[3]{y}$ near the point $(81, 125)$.

$$L(x, y) = f(81, 125) + f_x(81, 125)(x - 81) + f_y(81, 125)(y - 125) =$$

$$4 + \left(\frac{1}{2}(81)^{-\frac{1}{2}}\right)(x - 81) + \left(\frac{1}{3}(125)^{-\frac{2}{3}}\right)(y - 125) =$$

$$4 + \left(\frac{1}{18}\right)(x - 81) + \left(\frac{1}{75}\right)(y - 125).$$

$$L(81.2, 124.8) = 4 + \left(\frac{1}{18}\right)(.2) + \left(\frac{1}{75}\right)(-.2).$$

6. Each function matches exactly one of the pictures on the following pages, either a graph, or a set of level curves (for equally spaced values of f). Identify the picture that goes with each function.

(a) $f(x, y) = x^2 + 4y^2$

Solution: Figure 1 is the graph of f. (The level curves of f are ellipses, but if you graph one, you will see they are not the same as the ellipses of Figure 4.)

(b) $f(x, y) = x^2 + 2xy + y^2$

Solution: Figure 5 is the level curves of f. (We have $f(x, y) = (x + y)^2$, so the level curves of f are (pairs of) straight lines $x+y = \pm c$. The graph of f is steeper for larger values of $x + y$, so the lines are not evenly spaced as in Figure 6, but more closely spaced as $x + y$ is larger.)

(c) $f(x, y) = 4x^2 - 2y^2$

Solution: Figure 2 is the graph of f. (The level curves are hyperbolae. The intersections of the graph with the planes $y = 0$ and $x = 0$ are parabolae, one facing upward, and one facing downward.)

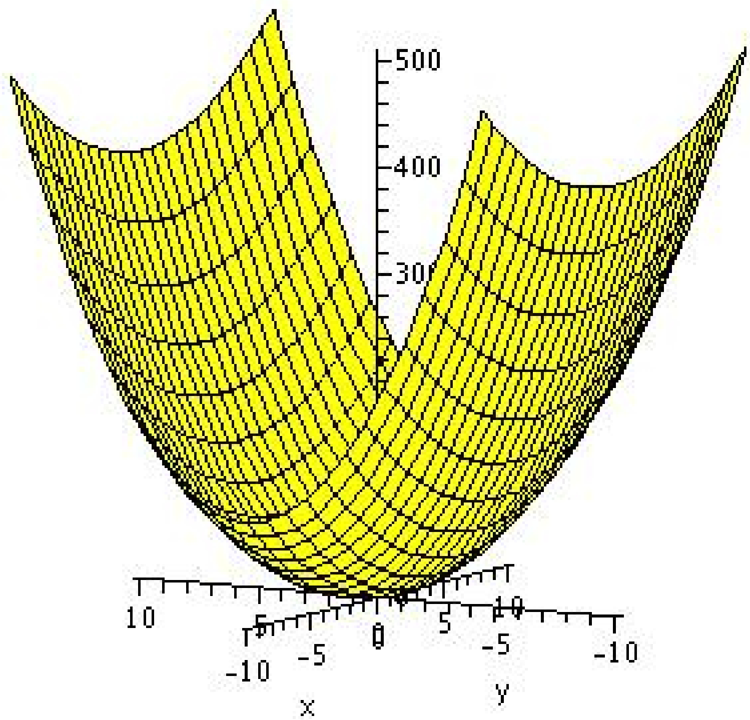

Figure 1: Graph of f.

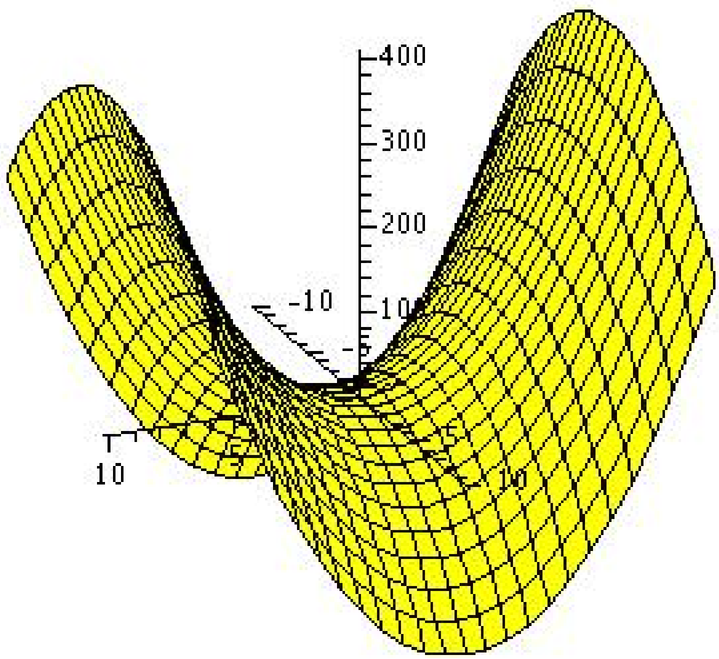

Figure 2: Graph of f.

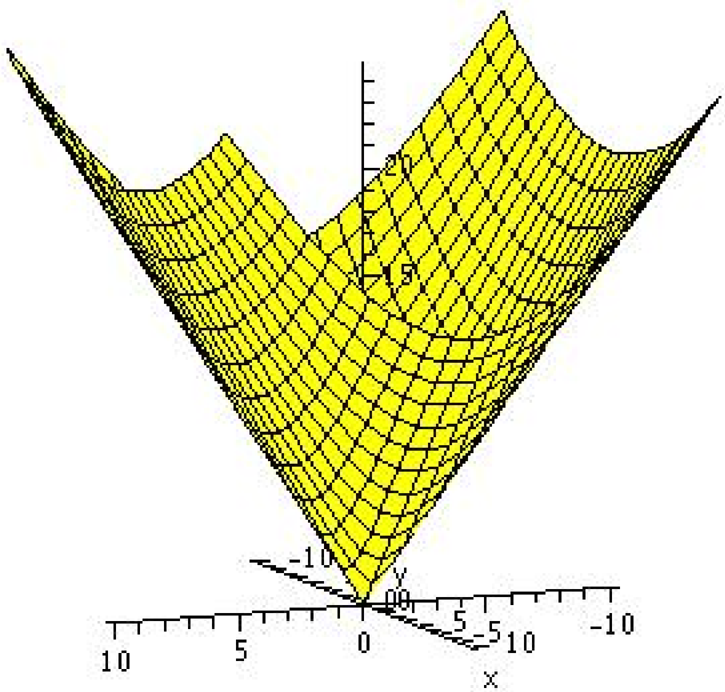

Figure 3: Graph of f.

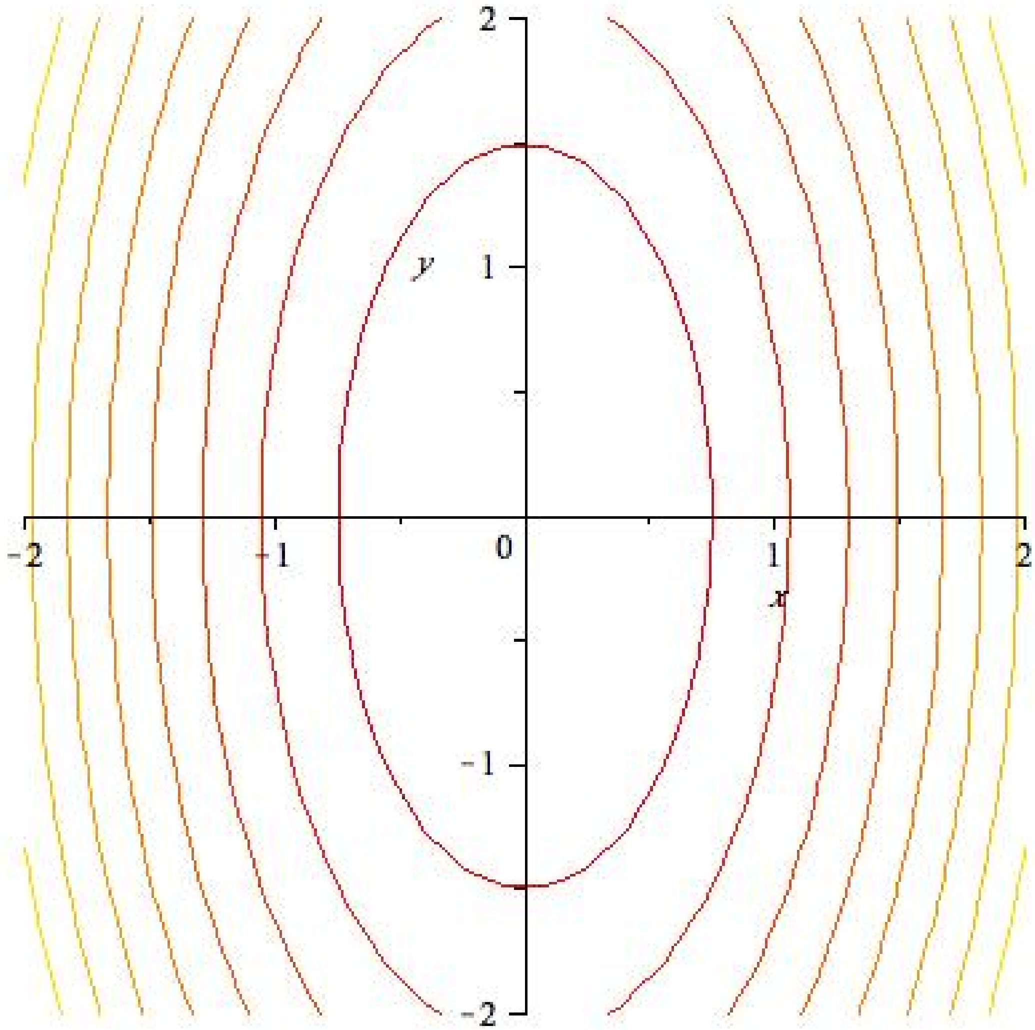

Figure 4: Level curves of f.

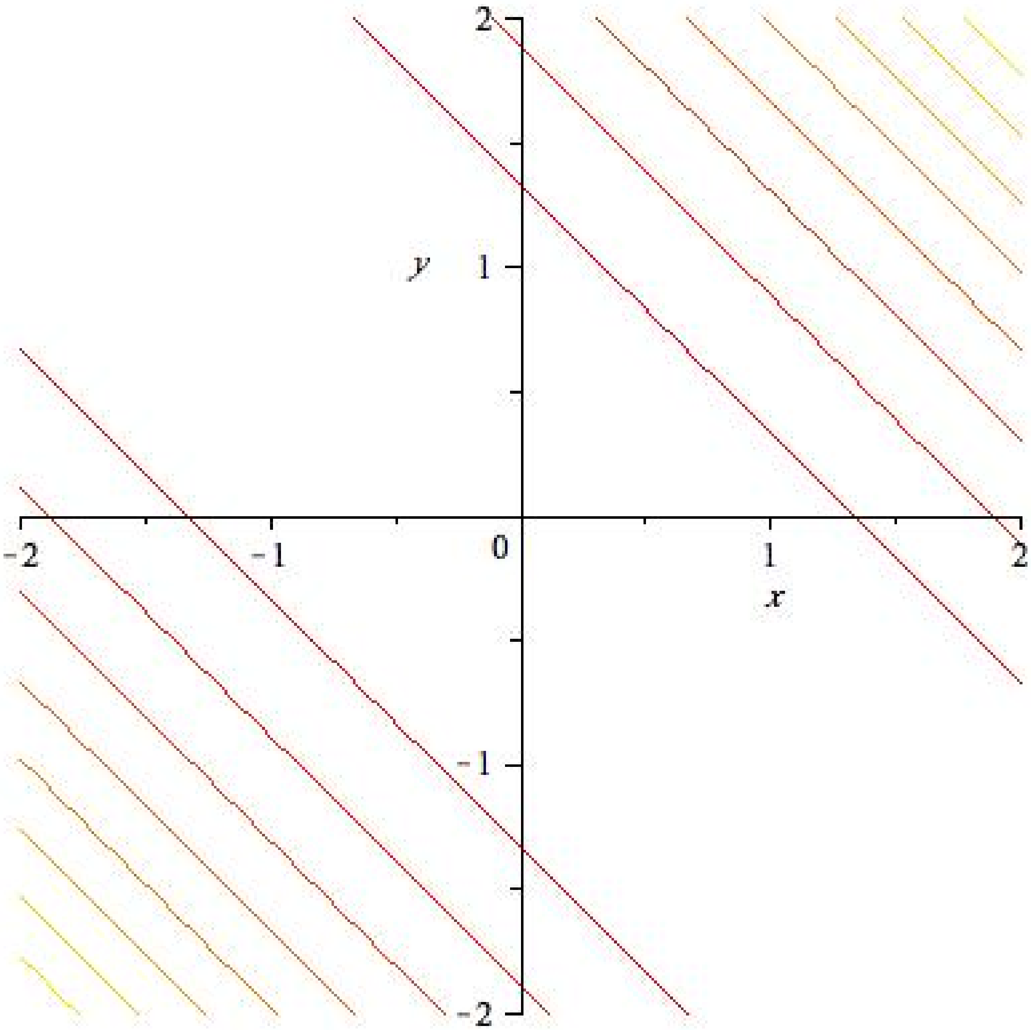

Figure 5: Level curves of f.

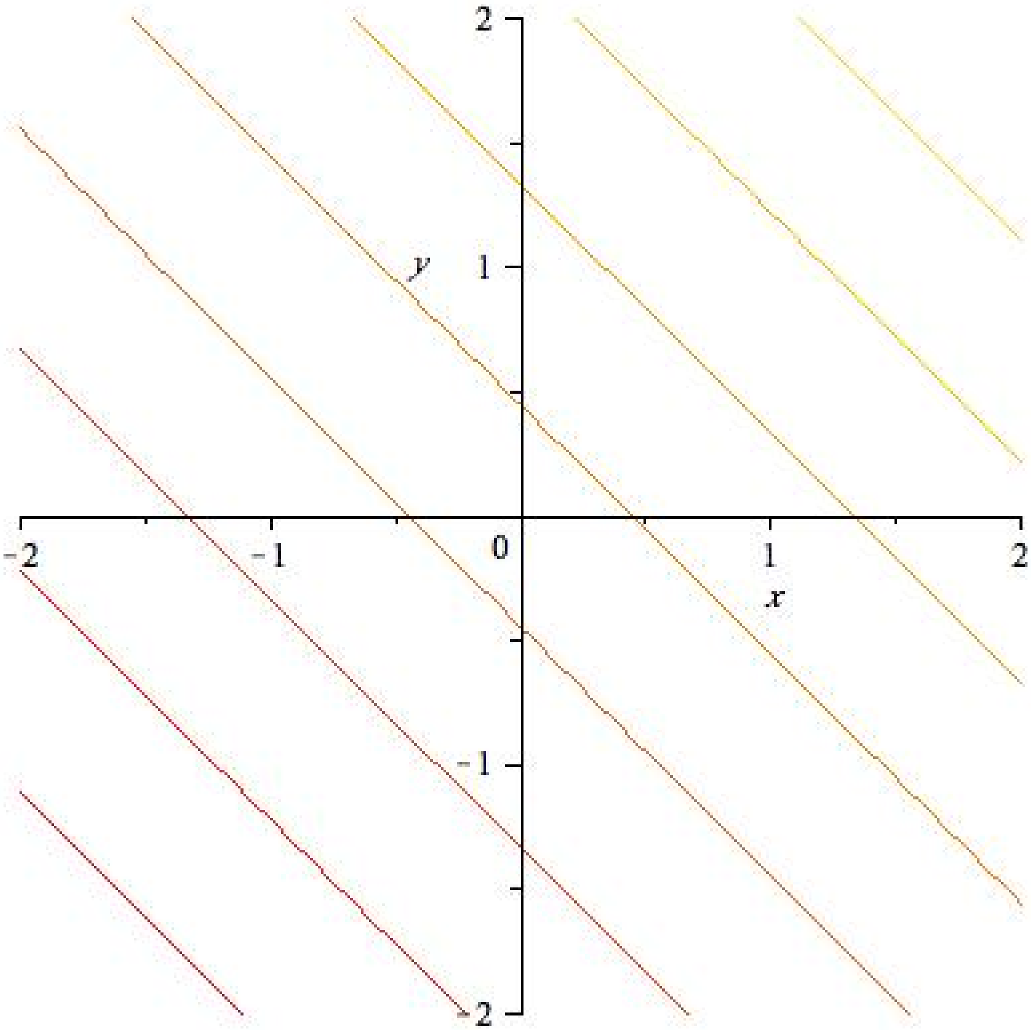

Figure 6: Level curves of f.

Other Practice Problems

1. Let $A = (2, 3, 2)$, $B = (3, 1, -1)$, and $C = (5, 2, 3)$.

   (a) Is the angle ABC acute, obtuse, or straight? Explain your answer.

   (b) Find the area of triangle ABC.

2. Determine whether the lines $\vec{r} = \langle 0,\ 1,\ 0\rangle + t\,\langle 1,\ 2,\ 2\rangle$ and $\vec{r} = \langle 2,\ 2,\ 6\rangle + t\,\langle 1,\ -1,\ 4\rangle$ intersect, and if so, find their point of intersection.

3. A moving particle has initial position $\vec{r}(0) = \hat{\imath} + \hat{\jmath} + \hat{k}$ and initial velocity $\vec{v}(0) = \hat{\imath} + \hat{k}$, and at time t has acceleration $\vec{a}(t) = -(\cos t)\hat{\imath} - (\sin t)\hat{\jmath} - (4t)\hat{k}$. Find the particle's position $\vec{r}(t)$ and velocity $\vec{v}(t)$ at time t.

4. Determine whether the limit exists, and if so, find its value.

   (a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3xy}{x^2 + y^2}$

   (b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3xy^2}{x^2 + y^2}$

5. Consider the ellipsoid $2x^2 + 3y^2 + z^2 = 18$.

   (a) Find an equation of the tangent plane to the surface of the ellipsoid at the point $(-1, 2, -2)$.

   (b) Determine parametric equations describing the normal line at the same point.

6. The temperature at a point (x, y, z) is given by

$$T(x, y, z) = 3x^2 + y^2 - z.$$

   A moving particle has position at time t given by $\vec{r}(t) = \langle 2e^{-6t},\ 2e^{-2t},\ 2t\rangle$.

   (a) Show that at all times, the particle is moving in the direction in which temperature decreases fastest.

   (b) Find the rate of change of the particle's temperature with respect to time when $t = 0$.