Math 11 Section 1
September 28, 2012
Sample Solutions

Now we know what "differentiable" means for a function $f : \mathbb{R}^n \to \mathbb{R}$.

> The function f is differentiable at the point $\vec{r}_0$ if and only if there is a linear function $L : \mathbb{R}^n \to \mathbb{R}$ whose graph is tangent to the graph of f at $\vec{r} = \vec{r}_0$.

This allows us to put all our tangent approximations into one form. First:

> If $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable, then its *derivative* (also, its *gradient*) is the vector of its partial derivatives:
>
> $$f'(x, y, \ldots) = \nabla f(x, y, \ldots) = \langle\, f_x(x, y, \ldots),\; f_y(x, y, \ldots),\; \ldots \rangle\,.$$

(1.) Let

$$f(x, y, z) = xe^y - y^2 z.$$

Find:

$f(3, 0, 2) = (3)(e^0) - (0^2)(2) = \boxed{3}$

$f'(x, y, z) = \left\langle \frac{\partial f}{\partial x},\, \frac{\partial f}{\partial y},\, \frac{\partial f}{\partial z} \right\rangle = \boxed{\left\langle e^y,\; xe^y - 2yz,\; -y^2 \right\rangle}$

$f'(3, 0, 2) = \boxed{\langle 1, 3, 0\rangle}$

Now for any function we have the same form for the tangent approximation near a point, the linear function with the same value and derivative at that point.

For $f : \mathbb{R} \to \mathbb{R}$, the single-variable version, we have the tangent approximation at x_0:

$$f(x) \approx L(x) = f(x_0) + f'(x_0)(x - x_0).$$

For $\vec{r} : \mathbb{R} \to \mathbb{R}^n$, a function parametrizing a curve, we have the tangent approximation at t_0:

$$\vec{r}(t) \approx \vec{L}(t) = \vec{r}(t_0) + \vec{r}\,'(t_0)(t - t_0).$$

For $\vec{f} : \mathbb{R}^n \to \mathbb{R}$, we have the tangent approximation at $\vec{r}_0 = (x_0, y_0, \ldots)$:

$$f(\vec{r}) \approx L(\vec{r}) = f(\vec{r}_0) + f'(\vec{r}_0)\cdot(\vec{r} - \vec{r}_0) = \boxed{f(x_0, y_0 \ldots) + f'(x_0, y_0, \ldots) \cdot \langle x - x_0,\; y - y_0,\; \ldots \rangle} =$$

$$f(x_0, y_0, \ldots) + \langle\, f_x(x_0, y_0, \ldots),\; f_y(x_0, y_0, \ldots),\; \ldots \rangle \cdot \langle x - x_0,\; y - y_0,\; \ldots \rangle =$$

$$f(x_0, y_0, \ldots) + (f_x(x_0, y_0, \ldots))\,(x - x_0) + (f_y(x_0, y_0, \ldots))\,(y - y_0) + \cdots$$

We can also express this as: If w is a function of $(x, y, \dots)$, then near $(x_0, y_0, \dots)$,

$$w \approx w_0 + \left(\frac{\partial w}{\partial x}\right)(x - x_0) + \left(\frac{\partial w}{\partial y}\right)(y - y_0) + \cdots .$$

(2.) Use the boxed formula to approximate the function of problem (1),

$$f(x, y, z) = xe^y - y^2 z,$$

near the point $(3, 0, 2)$. Put your answer in the form $f(x, y, z) \approx ax + by + cz + d$.

We use the boxed formula, with the point $(x_0, y_0, z_0) = (3, 0, 2)$, and the values for f and f' we found in problem (1).

$f(x, y, z) \approx L(x, y, z) = f(3, 0, 2) + f'(3, 0, 2) \cdot \langle x - 3, y - 0, z - 2\rangle$

$\approx 3 + \langle 1, 3, 0\rangle \cdot \langle x - 3, y - 0, z - 2\rangle = 3 + 1(x - 3) + 3(y - 0) + 0(z - 2) = \boxed{x + 3y}$

(3.) Find an approximate value for $(2.98)e^{-.01} - (-.01)^2(2.03)$.

Note that $(2.98)e^{-.01} - (-.01)^2(2.03) = f(2.98, -.01, 2.03)$.
Since $(2.98, -.01, 2.03)$ seems to be near $(3, 0, 2)$, we use the approximation we found in problem (2), $f(x, y) \approx x + 3y$, to get

$(2.98)e^{-.01} - (-.01)^2(2.03) = f(2.98, -.01, 2.03) \approx 2.98 + 3(-.01) = \boxed{2.95}$.