Math 11 Section 1
September 26, 2012
Sample Solutions

(1.) Last time we saw that $\frac{\partial f}{\partial x}(x_0, y_0)$ is the vertical slope of the intersection of the graph of f with the plane $y = y_0$, so that $\left\langle 1,\ 0,\ \frac{\partial f}{\partial x}(x_0, y_0)\right\rangle$ is a vector tangent to the graph of f at the point $(x_0,\ y_0,\ f(x_0, y_0))$. Similarly the vector $\left\langle 0,\ 1,\ \frac{\partial f}{\partial y}(x_0, y_0)\right\rangle$ is tangent to the graph of f at the point $(x_0,\ y_0,\ f(x_0, y_0))$.

Use this to find two vectors tangent to the graph of the function $f(x, y) = x^2 - xy + y^2$ at the point $(1,\ -1,\ 3)$.

$$\left\langle 1, 0, \frac{\partial f}{\partial x}\right\rangle = \langle 1, 0, 2x - y\rangle = \boxed{\langle 1, 0, 3\rangle}$$
$$\left\langle 0, 1, \frac{\partial f}{\partial y}\right\rangle = \langle 0, 1, -x + 2y\rangle = \boxed{\langle 0, 1, -3\rangle}$$

Find a vector normal to the graph of the function $f(x, y) = x^2 - xy + y^2$ at the point $(1,\ -1,\ 3)$.

$$\langle 1, 0, 3\rangle \times \langle 0, 1, -3\rangle = \boxed{\langle -3, 3, 1\rangle}$$

Find an equation for the plane tangent to the graph of the function $f(x, y) = x^2 - xy + y^2$ at the point $(1,\ -1,\ 3)$.

This is the plane containing the point $(1,\ -1,\ 3)$ with normal vector $\langle -3, 3, 1\rangle$. Its equation is $\langle -3, 3, 1\rangle \cdot \langle x - 1, y - (-1), z - 3\rangle = 0$, or $\boxed{-3x + 3y + z = -3}$.

(2.) Applying this same reasoning to any function f gives us the fact that if f has a tangent plane at $(x_0,\ y_0,\ f(x_0, y_0))$, then that tangent plane is given by

$$z = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0).$$

We can use this to get a linear tangent approximation to $f(x, y)$: If (x, y) is near (x_0, y_0), then

$$f(x, y) \approx L(x, y) = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0).$$

(This should look a lot like the tangent line approximation to a function $f(x)$.)

Use this method, applied to the function $f(x, y) = xe^y$ near the point $(2, 0)$, to approximate $(2.005)e^{-.01}$.

First we compute

$$\frac{\partial f}{\partial x}(x,y) = e^y \qquad \frac{\partial f}{\partial y}(x,y) = xe^y.$$

Using $(x_0, y_0) = (2,0)$, we get that if (x,y) is near $(2,0)$, then

$f(x,y) \approx L(x,y) = f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0) =$
$2e^0 + (e^0)(x-2) + (2e^0)(y-2) = 2 + 1(x-2) + 2(y-0)$.

$(2.005)e^{-.01} = f(2.005, -.01) \approx 2 + 1(2.005-2) + 2(-.01-0) = 2 + .005 - .02 = \boxed{1.985}$.

(3.) A theorem from the text tells us that the graph of $f(x,y)$ does have a tangent plane at $(x_0,\ y_0,\ f(x_0,y_0))$ if the partial derivatives of f are continuous on some disc containing (x_0, y_0).

(The graph does not have a tangent plane if the partial derivatives of f are undefined at (x_0, y_0). If the partial derivatives of f are defined at the point but not continuous on any disc containing the point, the graph may or may not have a tangent plane.)

Consider the function

$$f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0); \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Show that f is continuous at $(x,y) = (0,0)$.

We must show that $\lim\limits_{(x,y)\to(0,0)} f(x,y) = f(0,0) = 0$. From the definition of f,

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}.$$

We can analyze this limit in different ways; perhaps the easiest is to use polar coordinates:

$$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{r^2 \sin\theta\cos\theta}{r} = \lim_{(x,y)\to(0,0)} r\sin\theta\cos\theta.$$

Since the absolute value of $\sin\theta\cos\theta$ is at most 1, we have $-r \leq r\sin\theta\cos\theta \leq r$.

Since as $(x,y) \to (0,0)$ both $-r$ and r approach 0, by the Squeeze Theorem we have

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} r\sin\theta\cos\theta = 0.$$

This is what we needed to show.

Since the definition of f has a special case at $(0,0)$, to compute the partial derivatives of f at $(0,0)$ we will go back to the definition of partial derivative: $\frac{\partial f}{\partial x}(x_0, y_0)$ is found by setting y to be a constant y_0, differentiating the resulting function of x, and evaluating that derivative at x_0:

For example, if we set $y = 0$, then the definition of f tells us that

$$f(x,0) = \left[\begin{cases} \frac{0}{\sqrt{x^2}} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}\right] = 0.$$

$$f(x,0) = \underline{0} \qquad \frac{\partial f}{\partial x}(x,0) = \underline{0} \qquad \frac{\partial f}{\partial x}(0,0) = \underline{0}$$

$$f(0,y) = \underline{0} \qquad \frac{\partial f}{\partial y}(0,y) = \underline{0} \qquad \frac{\partial f}{\partial y}(0,0) = \underline{0}$$

Recall that

$$f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0); \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

If $(x,y) \neq (0,0)$ then $\frac{\partial f}{\partial x}(x,y) = \boxed{\frac{y^2}{(x^2+y^2)^{\frac{3}{2}}}}$

Show that $\frac{\partial f}{\partial x}$ is not continuous at $(0,0)$.

If $y = 0$ then $\frac{\partial f}{\partial x} = 0$, so as $(x,y) \to (0,0)$ along the x-axis, $\frac{\partial f}{\partial x}$ approaches 0.

If $x = 0$ and $y > 0$, then $\frac{\partial f}{\partial x} = 1$ so as $(x,y) \to (0,0)$ along the positive y-axis, $\frac{\partial f}{\partial x}$ approaches 1.

This shows that $\frac{\partial f}{\partial x}$ is not continuous at $(0,0)$.

Now f has partial derivatives at $(0,0)$ but they are not continuous there, so we do not know whether the graph of f has a tangent plane at $(0,0,0)$. Sketch the intersection of the graph of f with the plane $x = y$. Note that this plane contains the vertical z-axis and the horizontal line $x = y$, which intersect at $(0,0)$; you can start by putting them in your sketch.

If you can't picture immediately what this curve looks like, you might start by parametrizing it, using $x = t$. Hint: Your parametrizing function will have a different definition when $t \leq 0$ and when $t \geq 0$, because $\sqrt{t^2} = \pm t$.

If that still doesn't give you the picture, try differentiating your parametrizing function $\vec{r}$ to see what the tangent vectors look like at various points.

This intersection is given by $x = y$ and $z = \frac{x^2}{\sqrt{2x^2}} = \frac{x^2}{|x|\sqrt{2}} = \frac{|x|}{\sqrt{2}}$. It looks like the graph of $y = |x|$; it has a sharp corner at the origin.

This should tell you that the graph of f does not have a tangent plane at $(0,0,0)$.