Math 11 Section 1
September 21, 2012
Sample Solutions

Today we are going to talk, among other things, about limits of functions of two or more variables. We'll look at some examples, and then talk about the definition.

(1.) First we will consider the function

$$f(x,y) = \frac{x^3}{x^2+y^2}.$$

To start thinking about the limit

$$\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2},$$

first compute the following limits

$$\lim_{x\to 0} \frac{x^3}{x^2+0^2} = \boxed{0}$$

$$\lim_{y\to 0} \frac{0^3}{0^2+y^2} = \boxed{0}$$

In single-variable calculus we had left-hand limits and right-hand limits, as our argument approached the limiting value from the left or from the right. The two limits you just computed can be thought of as the limits of $f(x,y)$ as $(x,y)\to(0,0)$ along the line $y=0$ (the x-axis), and along the line $x=0$ (the y-axis).

You might think we know the limit of $f(x,y)$ as $(x,y)\to(0,0)$ now. But consider this next example:

(2.) Let

$$g(x,y) = \frac{xy}{x^2+y^2}.$$

Compute the limit of $g(x,y)$ as $(x,y)\to(0,0)$ along the x-axis and along the y-axis.

$$\lim_{(x,y)\to(0,0)} \frac{0}{x^2} = \boxed{0}$$

$$\lim_{(x,y)\to(0,0)} \frac{0}{y^2} = \boxed{0}$$

Now find the limit of $g(x,y)$ as $(x,y)\to(0,0)$ along the line $x=y$ by computing

$$\lim_{x\to 0} \frac{x\cdot x}{x^2+x^2}.$$

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{2x^2} = \boxed{\frac{1}{2}}$$

Since we get different limits as $(x, y) \to (0, 0)$ along different paths, as in the single-variable case of different right- and left-hand limits, we say

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} \text{ is } \textit{undefined.}$$

(3.) Let's go back to the function $f(x, y)$ of problem (1). We can't check the limit of $f(x, y)$ as $(x, y) \to (0, 0)$ along all possible lines, because there are infinitely many of them. (And it's even worse than that; we would also have to check curved paths.)

(a.) Show that for all $(x, y) \neq (0, 0)$, we have

$$|f(x, y)| \leq |x|.$$

Because $x^2 + y^2 \geq x^2$, it follows that if $x \neq 0$

$$|f(x, y)| = \frac{|x^3|}{x^2+y^2} \leq \frac{|x^3|}{x^2} = |x|.$$

Of course, if $x = 0$ then $f(x, y) = 0$, so in this case as well, $|f(x, y)| \leq |x|$.

(b.) Give a short argument (just a sentence or two) that

$$\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2} = 0.$$

Clearly, as (x, y) approaches $(0, 0)$, also x approaches 0. Since $f(x, y)$ is even smaller than x (in absolute value), it too much approach 0. [Later, we will be able to give a more formal argument using the Squeeze Theorem.]

(4.) Now define

$$h(x, y) = \begin{cases} 1 & \text{if } y = x^2 \text{ and } (x, y) \neq (0, 0); \\ 0 & \text{if } y \neq x^2 \text{ or } (x, y) = (0, 0). \end{cases}$$

That is, h has value 1 on the parabola $y = x^2$ except at the origin, and 0 everywhere else.

(a.) Let ℓ be any line through the origin. If ℓ is the x-axis, ℓ intersects the parabola $y = x^2$ only at the origin, so $h(x, y) = 0$ for all (x, y) on ℓ. If ℓ is not the x-axis, at how many points does ℓ intersect the parabola $y = x^2$?

Two, $x = 0$ and $x = a$.

What is the limit of $h(x, y)$ as $(x, y) \to (0, 0)$ along a line ℓ?

The value of h on this line is 0 except possibly at one point (a, a^2), so the limit is 0.

What is the limit of $h(x, y)$ as $(x, y) \to (0, 0)$ along the parabola $y = x^2$?

Except at the point $(0, 0)$, the value of h is 1, so the limit is 1.

This example illustrates that, even if we could check the limit as $(x, y) \to (0, 0)$ along every line, that still wouldn't be enough to determine the limit.

(5.) Now consider the function f of problem (1) again. According to the formal definition of limit,

$$\lim_{(x,y)\to(0,0)} f(x, y) = 0$$

means that for any positive number $\varepsilon > 0$, no matter how small, there is a corresponding positive distance $\delta > 0$ with the property that whenever (x, y) is closer to $(0, 0)$ than distance δ (and is not equal to $(0, 0)$), then the value $f(x, y)$ is within ε of 0,

$$0 < |\langle x, y\rangle - \langle 0, 0\rangle| \leq \delta \implies |f(x, y) - 0| \leq \varepsilon.$$

(Less formally, we can make $f(x, y)$ as close as we like to 0 (within ε) by making (x, y) close enough to $(0, 0)$ (within δ).)

Show this is true if we take $\delta = \varepsilon$. That is, show that

$$0 < |\langle x, y\rangle - \langle 0, 0\rangle| \leq \varepsilon \implies |f(x, y) - 0| \leq \varepsilon.$$

(Problem (3a) might be useful.)

Assume that

$$0 < |\langle x, y\rangle - \langle 0, 0\rangle| \leq \delta = \varepsilon.$$

Using the result of problem (3):

$$|f(x, y) - 0| = |f(x, y)| \leq |x| = \sqrt{x^2} \leq \sqrt{x^2 + y^2} = |\langle x, y\rangle - \langle 0, 0\rangle| \leq \delta = \varepsilon.$$