Math 11 Section 1
September 19, 2012
Sample Solutions

For this set of questions, let

$$\vec{r}(t) = \langle cos(t),\ sin(t),\ t\rangle$$

be the position at time t of a moving object. All the questions are about this specific example.

(1.) Compute the velocity, the speed, and the unit tangent vector to the curve parametrized by $\vec{r}$ at time t.

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = \vec{r}\,'\ (t) = \boxed{\langle -\sin(t), \cos(t), 1\rangle}$$

$$\text{speed} = \frac{ds}{dt} = |\vec{r}\,'(t)| = \sqrt{\cos^2(t) + \sin^2(t) + 1} = \boxed{\sqrt{2}}$$

$$\vec{T}(t) = \boxed{\frac{1}{\sqrt{2}}\langle -\sin(t), \cos(t), 1\rangle}$$

(2.) Compute the derivative of the unit tangent vector with respect to time.

$$\frac{d\vec{T}}{dt} = \vec{T}\,'(t) = \boxed{\frac{1}{\sqrt{2}}\langle -\cos(t), -\sin(t), 0\rangle}$$

Verify that in this example, $\vec{T}\,'(t)$ is orthogonal to $\vec{T}$.

Check that the vectors are orthogonal by checking that their dot product is zero:
$\vec{T}\,'(t)\cdot\vec{T}(t) = \frac{1}{\sqrt{2}}\langle -\cos(t), -\sin(t), 0\rangle \cdot \frac{1}{\sqrt{2}}\langle -\sin(t), \cos(t), 1\rangle = 0.$

(3.) Compute the magnitude of $\vec{T}\,'(t)$, and a unit vector $\vec{N}(t)$ in the direction of $\vec{T}\,'(t)$. This vector $\vec{N}$ is called the unit normal vector to γ.

$$\left|\vec{T}\,'(t)\right| = \left|\frac{1}{\sqrt{2}}\langle -\cos(t), -\sin(t), 0\rangle\right| = \boxed{\frac{1}{\sqrt{2}}}$$

$$\vec{N}(t) = \boxed{\langle -\cos(t), -\sin(t), 0\rangle}$$

(4.) Compute the cross product of $\vec{T}(t)$ and $\vec{N}(t)$. This vector $\vec{B}$ is called the unit binormal vector to γ.

$$\vec{B}(t) = \vec{T}(t)\times\vec{N}(t) = \frac{1}{\sqrt{2}}\langle -\sin(t), \cos(t), 1\rangle \times \langle -\cos(t), -\sin(t), 0\rangle =$$

$$\boxed{\frac{1}{\sqrt{2}}\langle \sin(t), -\cos(t), 1\rangle}$$

Explain why $\vec{B}(t)$ is automatically a unit vector.

By the geometric interpretation of the cross product, the length of $\vec{T} \times \vec{N}$ is the product of the length of $\vec{T}$, the length of $\vec{N}$, and the cosine of the angle θ between them. Because $\vec{T}$ and $\vec{N}$ are unit vectors, their lengths are 1. Because they are orthogonal, $\theta = \frac{\pi}{2}$, and $\cos(\theta)$ is also 1. Therefore, $|\vec{B}| = (1)(1)(1) = 1$.

(4.) The arclength of the curve γ parametrized by a function $\vec{f}$ (with no retracing of the curve) between points $\vec{f}(a)$ and $\vec{f}(b)$ is the integral of the speed between times a and b,

$$\int_\gamma ds = \int_a^b \frac{ds}{dt}\, dt.$$

Compute the arclength of one loop of the helix parametrized by $\vec{r}$.

$$\int_0^{2\pi} \frac{ds}{dt}\, dt = \int_0^{2\pi} \sqrt{2}\, dt = \boxed{2\sqrt{2}\pi}$$

The variable s represents *arclength*, or distance along the curve. If you want to be specific, you can pick any time a, and let $s(b)$ be the distance traveled between time a and time b for $b > a$, and the negative of the distance traveled between time b and time a for $b < a$. That is,

$$s(b) = \int_a^b \frac{ds}{dt}\, ds.$$

This gives us many arclength functions, but they differ by a constant (the distance along the curve between the different choices of a), so they all have the same derivative.

(5.) Use the form of the Chain Rule[1]

$$\frac{d\vec{T}}{dt} = \frac{d\vec{T}}{ds}\frac{ds}{dt}$$

to compute the rate of change of the unit tangent vector with respect to arclength (in our example).

$$\frac{d\vec{T}}{ds} = \frac{\frac{d\vec{T}}{dt}}{\frac{ds}{dt}} = \frac{\frac{1}{\sqrt{2}}\langle -\cos(t), -\sin(t), 0\rangle}{\sqrt{2}} = \boxed{\left\langle -\frac{\cos(t)}{2}, -\frac{\sin(t)}{2}, 0\right\rangle}$$

[1]NOTE: You should know the basic facts given in the textbook, such as the rules for vector arithmetic (p. 819), dot product (p. 825), cross product (p. 836), and derivative (p. 874), even though we haven't listed them all on the blackboard. This doesn't mean you have to memorize these lists; it means you should be able to recall and use any of these facts if you need to.

(6.) Compute the magnitude of the rate of change of the unit tangent vector with respect to arclength (in our example). This is called the *curvature*, and denoted by κ (kappa).

$$\kappa = \left|\frac{d\vec{T}}{ds}\right| = \boxed{\frac{1}{2}}$$

(7.) Verify that for this example, we can write the acceleration as

$$\vec{a} = \frac{d}{dt}\left(\frac{ds}{dt}\right)\vec{T} + \left(\frac{ds}{dt}\right)^2 \kappa\vec{N}.$$

We verify this by evaluating the left and right sides of the equation, to show they are equal. For the left-hand side, we differentiate velocity (which we have already found).

$$\vec{a} = \frac{d}{dt}\langle -\sin(t), \cos(t), 1\rangle =$$

$$\langle -\cos(t), -\sin(t), 0\rangle$$

For the right-hand side, we use the values of $\frac{ds}{dt}$, κ, and $\vec{N}$ we have already computed.

$$\frac{d}{dt}\left(\frac{ds}{dt}\right)\vec{T} + \left(\frac{ds}{dt}\right)^2 \kappa\vec{N} = \left(\frac{d}{dt}\left(\sqrt{2}\right)\right)\vec{T} + \left(\sqrt{2}\right)^2\frac{1}{2}\vec{N} = \vec{N} = 0(\vec{T}) + 1(\vec{N}) = \vec{N} =$$

$$\langle -\cos(t), -\sin(t), 0\rangle$$

The two vectors

$$\vec{a}_T = \left(\frac{d^2s}{dt^2}\right)\vec{T} \qquad \vec{a}_N = \left(\frac{ds}{dt}\right)^2 \kappa\vec{N}$$

are the *tangential* and *normal* components of the acceleration. The magnitude of the tangential component, $\frac{d^2s}{dt^2}$, is the rate of change of the speed, and is called the *linear acceleration.* If you have taken physics, and recall that force equals mass m times acceleration $\vec{a}$, you may recognize the magnitude of the normal component of the force (thinking $\kappa = \frac{1}{R}$, where R is the radius of a circle with curvature κ) as the magnitude of the *centripetal force.*