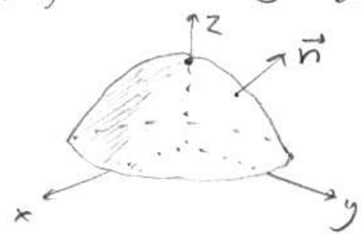
Barnett
11/19/10.

MATH 11 WORKSHEET : Surface Integrals

A) Let S be paraboloid $z = 1 - x^2 - y^2$ above the unit disc, $\vec{n}$ points up.

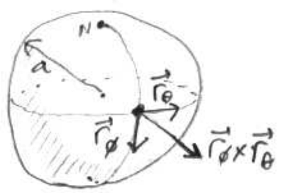


Compute $I = \iint_S (x, y, -x^2) \cdot d\vec{S}$

Hint: use $\left.\begin{array}{l} x = u \\ y = v \\ z = g(u,v) \end{array}\right\}$ first get $\vec{r}_u \times \vec{r}_v = \ldots$ [use lecture formula]

what is domain D?

B) Let S be the sphere of radius a with $\vec{n}$ pointing outwards.
Parametrize by ϕ, θ & find the 'directed area factor' $\vec{r}_\phi \times \vec{r}_\theta$:



$\vec{r}(\phi, \theta) = (\quad , \quad , \quad)$

so $\vec{r}_\phi = (\quad , \quad , \quad)$

$\vec{r}_\theta = (\quad , \quad , \quad)$

so $\vec{r}_\phi \times \vec{r}_\theta = (\quad , \quad , \quad)$

Simplify, then write as $\vec{r}_\phi \times \vec{r}_\theta =$ (some func of θ, ϕ) $\vec{r}$ ← hint: factor out $\vec{r}$ from above

C) Use this to find $\iint_S z^2 \, dS$:

from B) →

· SOLUTIONS ·

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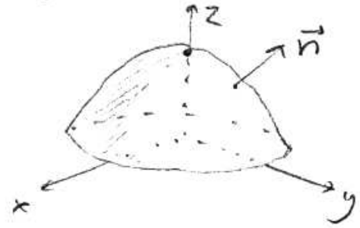
A) Let S be paraboloid $z = 1 - x^2 - y^2$ above the unit disc, $\vec{n}$ points up.



Compute $I = \iint_S (x, y, -x^2) \cdot d\vec{S}$ ← a vector surf. integral

Hint: use $x = u$, $y = v$, $z = g(u,v)$ ← what is domain D?

first get $\vec{r}_u \times \vec{r}_v = (-g_x, -g_y, 1)$ ← [use lecture formula]

$= (+2x, +2y, 1)$ where $g(x,y) = 1 - x^2 - y^2$

or $(2u, 2v, 1)$ if you prefer u,v parameters

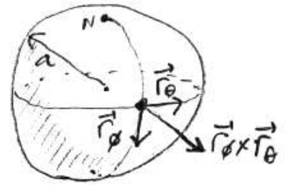
$\vec{F} \cdot \vec{r}_u \times \vec{r}_v = (u, v, -u^2) \cdot (2u, 2v, 1) = u^2 + 2v^2$

$$I = \iint_{\substack{\text{unit}\\ \text{disc}}} (u^2 + 2v^2)\, \underset{\text{eg } du\,dv}{dA} \overset{\text{polar}}{=} \int_0^{2\pi}\int_0^1 (r^2\cos^2\theta + 2r^2\sin^2\theta)\, r\,dr\,d\theta = \overbrace{\int_0^1 r^3 dr}^{1/4} \cdot \overbrace{\int_0^{2\pi} \cos^2\theta + 2\sin^2\theta\, d\theta}^{3\pi}$$

$= 3\pi/4$.

B) Let S be the sphere of radius a with $\vec{n}$ pointing outwards.

Parametrize by ϕ, θ & find the 'directed area factor' $\vec{r}_\phi \times \vec{r}_\theta$:



$\vec{r}(\phi, \theta) = (a\sin\phi\cos\theta, a\sin\phi\sin\theta, a\cos\phi)$

so $\vec{r}_\phi = (a\cos\phi\cos\theta, a\cos\phi\sin\theta, -a\sin\phi)$

$\vec{r}_\theta = (-a\sin\phi\sin\theta, a\sin\phi\cos\theta, 0)$

so $\vec{r}_\phi \times \vec{r}_\theta = (-a^2\sin^2\phi\cos\theta, -a^2\sin^2\phi\sin\theta, a^2\cos\phi\sin\phi\, \underbrace{(\cos^2\theta + \sin^2\theta)}_{1})$

Simplify, then write as $\vec{r}_\phi \times \vec{r}_\theta = (\text{some func of } \theta, \phi)\, \vec{r}$ ← hint: Factor out $\vec{r}$ from above.

$= (a\sin\phi)\vec{r} = a^2\sin\phi\, \vec{n}$ ← unit normal.

$dS = |\vec{r}_\phi \times \vec{r}_\theta|\, d\phi\, d\theta = a^2\sin\phi\, d\phi\, d\theta$ (from B) $z = a\cos\phi$

C) Use this to find $\iint_S z^2\, dS = \int_0^{2\pi}\int_0^{\pi} a^2\cos^2\phi \cdot a^2\sin\phi\, d\phi\, d\theta = a^4\int_0^{2\pi} d\theta \cdot \int_0^{\pi}\cos^2\phi\sin\phi\, d\phi$

scalar field!

$= \frac{4\pi a^4}{3}$