Barnett

→ SOLUTIONS ~ — .

Your name:

Instructor (please circle): Barnett Van Erp

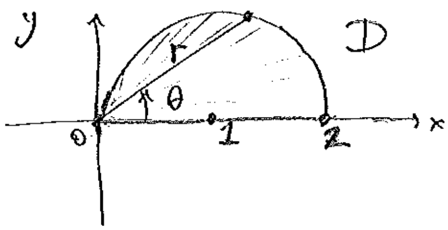
Math 11 Fall 2010: written part of HW6 (due Wed Nov 3)
Please show your work. No credit is given for solutions without justification.

(1) [8 points] Evaluate the following integral by changing to polar coordinates,

$$\int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{1}{\sqrt{x^2+y^2}}\, dy\, dx.$$

y limits 0 and $\sqrt{2x-x^2}$ $\Rightarrow$ curve $y^2 = 2x - x^2 = -(x-1)^2 + 1$

$(x-1)^2 + y^2 = 1$ circle.



However, you want polar about the origin (don't be distracted).

Sub. $\left.\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}\right\} \xrightarrow{\text{curve is}}$ $r^2\sin^2\theta = 2\cos\theta - r^2\cos^2\theta$

ie $r^2\underbrace{(\sin^2 + \cos^2)}_{1} = 2r\cos\theta$

$\Rightarrow r = 0$ or $r = 2\cos\theta$. (petal)

$$I = \int_0^{\pi/2}\int_0^{2\cos\theta} \frac{1}{r}\cdot r\,dr\,d\theta = \int_0^{\pi/2} \underbrace{\int_0^{2\cos\theta} 1\,dr}_{\to\, 2\cos\theta}\, d\theta$$

$$= 2\sin\theta\Big|_0^{\pi/2} = 2.$$

(2) [10 points] Let E be the solid region bounded by the surface $y = x^2$ and the two planes $z = 0$ and $y + z = 1$.

2 (a) Explain, without calculating the integral, why the value of the triple integral $\iiint_E z\,dV$ must be less than the volume of the solid E.

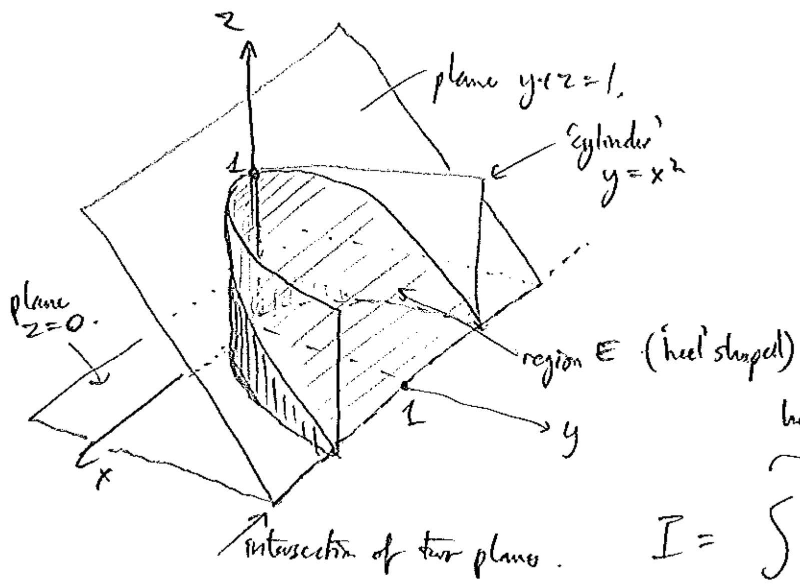
$f(x,y,z) = z \quad \le \quad g(x,y,z) = 1$ everywhere in E,

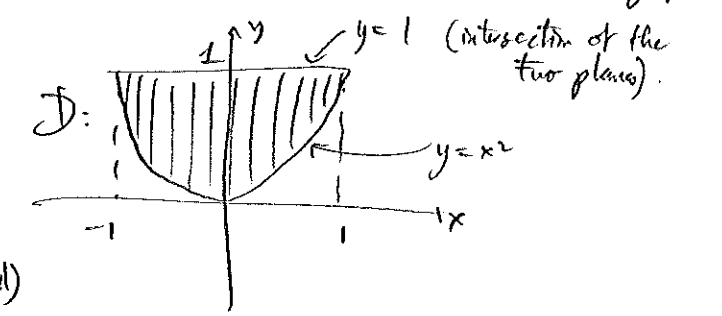
so $\iiint_E f\,dV \le \underbrace{\iiint_E g\,dV}_{\text{volume of } E\text{: the triple integral of } 1.}$ (the 3d analog of Eq. 9, Sec 16.1).

QED.

8. (b) Evaluate the triple integral $\iiint_E z\,dV$.

since z limits given vs. (x,y), Type 1. project 'shadow' down z onto xy plane:





$$I = \underbrace{\int_{-1}^{1}\int_{x^2}^{1}}_{\text{handles } D} \overbrace{\int_0^{1-y}}^{\text{upper plane } z=1-y} z\,dz\,dy\,dx$$

$$\int_0^{1-y} z\,dz = \frac{z^2}{2}\Big|_{z=0}^{z=1-y} = \frac{1}{2}(1-y)^2$$

$$I = \int_{-1}^{1}\underbrace{\int_{x^2}^{1} \tfrac{1}{2}(1-y)^2\,dy}\,dx$$

$$\to -\frac{1}{6}(1-y)^3\Big|_{y=x^2}^{y=1} = \frac{1}{6}(1-x^2)^3 = \frac{1}{6}(1-3x^2+3x^4-x^6)$$

$$I = \frac{1}{6}\left(x - x^3 + \frac{3}{5}x^5 - \frac{1}{7}x^7\right)\Big|_{-1}^{1} = \frac{1}{6}\left(2 - 2 + \frac{3\cdot 2}{5} - \frac{2}{7}\right) = \frac{1}{3}\left(\frac{21-5}{35}\right)$$

$$= \frac{16}{105}$$

(3) [8 points] Consider the iterated integral

$$\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_{-1}^{x} f(x,y,z)\,dz\,dx\,dy.$$

z first (Type 1)

x first (Type II)

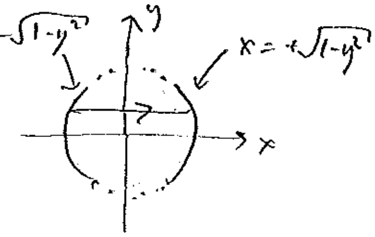
Rewrite this integral as an equivalent iterated integral in the form

$$\int\int\int f(x,y,z)\,dy\,dx\,dz.$$

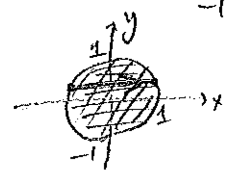
y first (Type 3)

inner integral is $z=-1$ to $z=x$

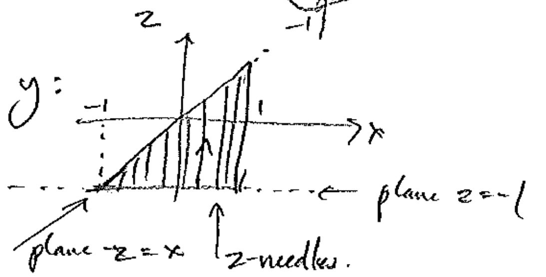
Shadow domain $\mathcal{D}$ in xy plane is $\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \cdots\, dx\,dy$

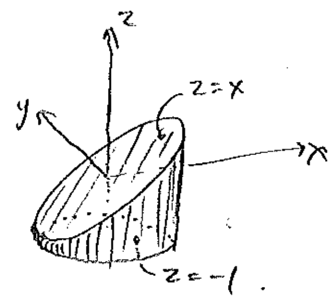


is unit disc



front view along y: (xz plane).



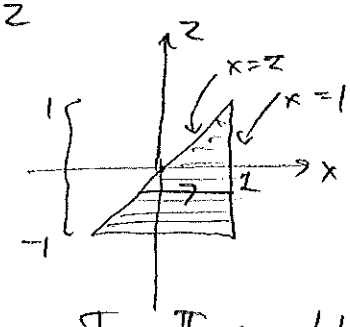


badly-chopped carrot shape. (truncated cylinder).

use this domain $\mathcal{D}'$ for y-first integral:

$$\int_{-1}^{1}\int_{z}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(x,y,z)\,dy\,dx\,dz$$

front & back cylinder surfaces depend only on x.



Type II approach to $\mathcal{D}'$.