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→ SOLUTIONS ←

Your name:

Instructor (please circle): Barnett Van Erp

Math 11 Fall 2010: written part of HW3 (due Wed Oct 13)
Please show your work. No credit is given for solutions without justification.

(1) [10 points] For each of the limits in items (a) and (b) below, either find the limit and prove that it is what you claim it is, or else prove that it does not exist.

(a) [3 points]

$$\lim_{(x,y)\to(0,0)} \frac{x^5 + x^2y^2}{x^4 + y^4}$$

try. x-axis $(y=0)$: $\lim_{x\to 0} \frac{x^5+0}{x^4+0} = 0$

y-axis $(x=0)$: $\lim_{y\to 0} \frac{0+0}{0+y^4} = 0$

try $y = mx$ (general slope m): $\lim_{x\to 0} \frac{x^5 + m^2x^4}{x^4+m^4x^4} = \lim_{x\to 0} \frac{x+m^2}{1+m^2} = \frac{m^2}{1+m^2}$

This varies with m, so limit doesn't exist.

(b) [3 points]

$$\lim_{(x,y)\to(0,0)} \frac{(x+y)^3}{x^2+y^2}$$

We try $y=0$, $x=0$, $y=mx$ as above and find all give limit of 0, so suspect (but don't yet know for sure) that lim exists. To prove it: sub $x = r\cos\theta$, $y = r\sin\theta$.

$$f(r,\theta) = \frac{(r\cos\theta + r\sin\theta)^3}{r^2\cos^2\theta + r^2\sin^2\theta} = \frac{r^3}{r^2}\frac{(\cos\theta+\sin\theta)^3}{1}$$

Fix r. Find $\max_\theta f(r,\theta) \le r\frac{(1+1)^3}{1} \le 8r$

$\min_\theta f(r,\theta) \ge r\frac{(-1-1)^3}{1} \ge -8r$.

Both tend to 0 as $\lim_{r\to 0}$, so limit exists & is zero.

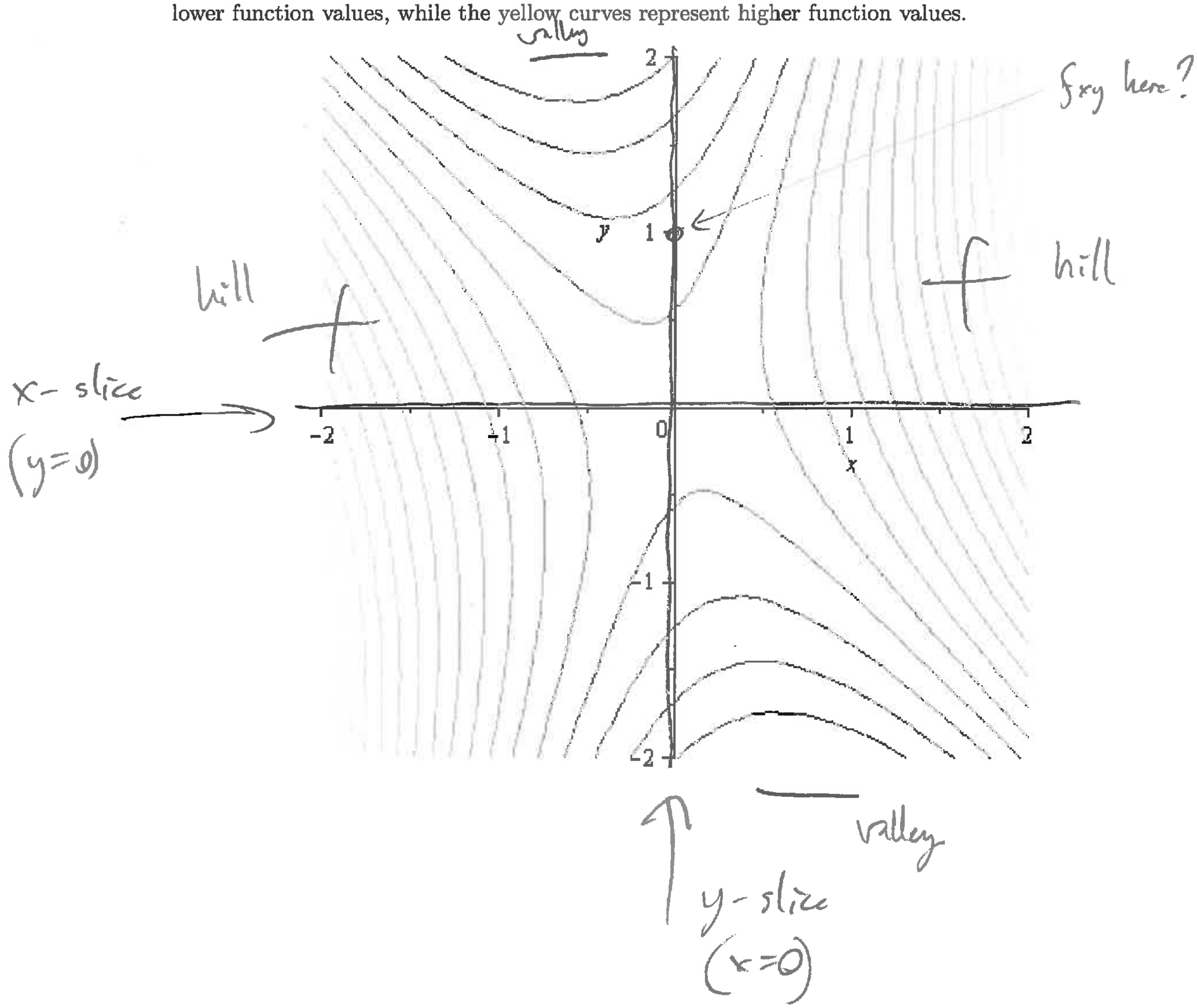
1

(c) [4 points] Find all points where the function $f(x, y)$ defined below is continuous. Justify your answer.

$$f(x,y) = \begin{cases} \cos(xy) & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

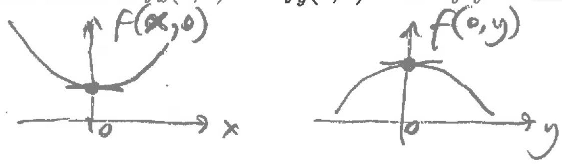
xy is continuous in all of $\mathbb{R}^2$, and $\cos(\cdot)$ is continuous for arguments in $\mathbb{R}$, so $\cos(xy)$ is continuous in all of $\mathbb{R}^2$. However at $(0,0)$, $\lim_{(x,y)\to(0,0)} f(x,y) = \cos 0 = 1$ but $f(0,0) = 0$ so f continuous for all $(x,y) \neq (0,0)$.

(2) [9 points] Below is part of the contour map for a two-variable function $f(x, y)$. The values of $f(x, y)$ for which the level curves are depicted are at chosen at constant intervals. The red curves represent lower function values, while the yellow curves represent higher function values.



(a) [2 points] What seem to be the values of $f_x(0,0)$ and $f_y(0,0)$? Justify your answer.

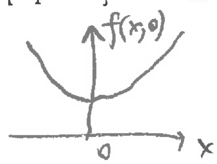
Plot slices:



In both slices, zero appears to be a place of zero slope.

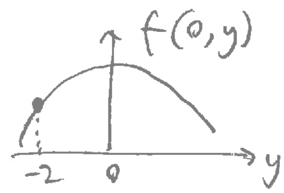
$\Rightarrow f_x(0,0) = f_y(0,0) = 0.$

(b) [2 points] What is the sign of the second derivative $f_{xx}(1,1)$? Justify your answer.



slice in x shows concave up. $\Rightarrow f_{xx} > 0$

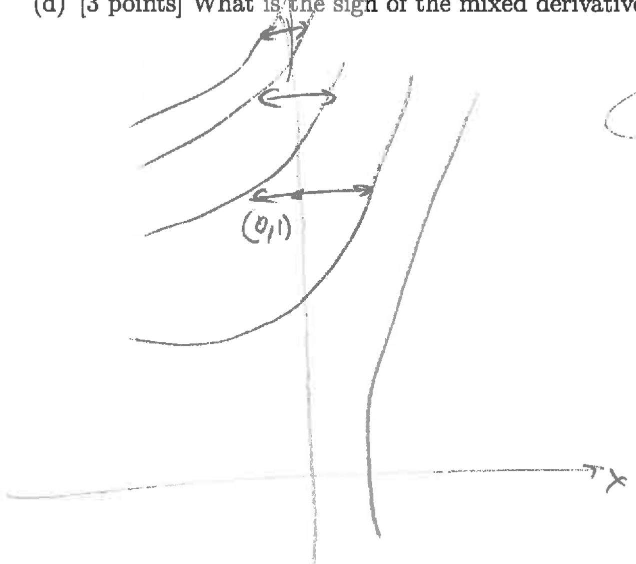
(c) [2 points] What is the sign of the second derivative $f_{yy}(0,-2)$? Justify your answer.



y-slice is concave down at $y=-2$

$\Rightarrow f_{yy} < 0.$

(d) [3 points] What is the sign of the mixed derivative f_{xy}(~~[illegible]~~ (0,1))? Justify your answer.



$\propto \frac{1}{f_x}$

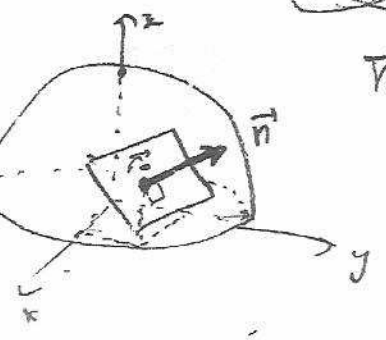
horizontal separation of contours decreases with increasing y

$\Rightarrow \frac{\partial}{\partial y} f_x = f_{xy} > 0$

if you're sneaky, you can realise that $\vec{n} = \vec{r}$ for sphere and save yourself lots of effort! :)

(3) [10 points] For items (a), (b), (c) below we let $f(x, y) = \sqrt{49 - x^2 - y^2}$. The graph of this function is a hemisphere.

(a) [3 points] Find an equation for the tangent plane to the graph of the function $f(x, y)$ at the point $(6, 3, 2)$. $\leftarrow \vec{r_0}$



$\vec{n}$ = normal of plane is $(-f_x, -f_y, 1)$

$f_x = -2x \cdot \frac{1/2}{\sqrt{49 - x^2 - y^2}}$, $f_y = -2y \cdot \frac{1/2}{\sqrt{49-x^2-y^2}}$

chain rule in 1 var.

so $\vec{n} = \left(\frac{+x}{\sqrt{49-x^2-y^2}}, \frac{+y}{\sqrt{49-x^2-y^2}}, 1\right) \overset{\vec{r}=\vec{r_0}}{=} \left(\frac{6}{2}, \frac{3}{2}, 1\right) = \left(3, \frac{3}{2}, 1\right)$

$\vec{n} \cdot \vec{r} - \vec{n} \cdot \vec{r_0} = 3x + \frac{3}{2}y + z - \frac{49}{2} = 0$

(b) [3 points] Calculate the angle between the xy plane and the tangent plane that you found in part (a).

$\vec{a} = (0, 0, 1)$

$\vec{b} = 2\vec{n} = (6, 3, 2)$ $\leftarrow$ doubling $\vec{n}$ has no effect yet avoids horrible fractions!

$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|\,|\vec{b}|} = \frac{2}{1 \cdot \sqrt{6^2 + 3^2 + 2^2}} = \frac{2}{7}$

$\theta = \cos^{-1}\frac{2}{7}$

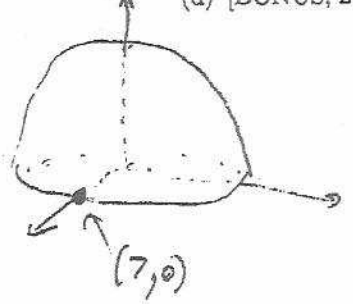
(c) [2 points] Is the function $f(x, y)$ differentiable at $(x, y) = (6, 3)$? Justify your answer.

f is composition of elementary functions, so is differentiable in interior of domain of definition $\quad 49 - x^2 - y^2 = 4$

so the $\sqrt{\cdot}$ func is not at edge of its domain

$\Rightarrow$ yes.

(d) [BONUS, 2 points] Do you think $f(x, y)$ is differentiable at $(x, y) = (7, 0)$? Explain your answer.



means f_x & f_y continuous at (x,y) & exist in some neighborhood of (x,y) [Thm 8, Sec 15.4]

However $f_x(7,0) = \frac{-x}{\sqrt{49-x^2-y^2}} = \frac{-7}{\sqrt{0}}$ doesn't even exist.

$\Rightarrow$ not differentiable there.