V63.0123-1 : Calculus III. Sample Midterm2 Answers

1. [12 points]

   Let $f(x, y) = \frac{x}{y} + \frac{y}{x}$.

   (a) $D = \{(x, y)|x \neq 0, y \neq 0\}$

   (b) $\nabla f(x, y) = \left(\frac{1}{y} - \frac{y}{x^2}, \frac{1}{x} - \frac{x}{y^2}\right)$.

   (c) $\nabla f \cdot \mathbf{r} = 0$ for all (x, y), by using $\mathbf{r} = (x, y)$. In other words $\left(\frac{1}{y} - \frac{y}{x^2}\right) x + \left(\frac{1}{x} - \frac{x}{y^2}\right) y$ simplifies to zero. Lengths are not zero $\Rightarrow$ perpendicular.

   (d) No. If approach along $y = x$, lim is $1 + 1 = 2$. If approach along x-axis or y-axis, lim is infinite.

2. [10 points]

   Treat as Type 1: for given (x, y), limits on z integral are 0 and $1 - x$. Remaining shadow region in xy plane has bounds $x = y^2$ and $x = 1$, treat as Type II is easiest. Ans:

$$\int_{-1}^{1} \int_{y^2}^{1} \int_{0}^{1-x} x \; dz \, dx \, dy = 1/3 - 1/5 + 2/21 = 8/35. \tag{1}$$

3. [8 points]

   Domain is semicircle in 1st and 4th quadrants. Assume density, our function $f(x, y)$ = 1. Then m = Area = half that of circle = $\pi/2$. Also $x = r \cos \theta$.

$$\bar{x} = \frac{1}{m} \int_{-\pi/2}^{\pi/2} \int_{0}^{1} r \cos \theta \; r dr \, d\theta = \frac{4}{3\pi}. \tag{2}$$

4. [10 points]

   Tree has f at top, x, y, z in middle, s, t as each sub-branch (apart from z which only has sub-branch t).

   $\frac{\partial f}{\partial s} = (y + z)t + (x + z)te^{st} \to 3 \quad$ at $\quad (s, t) = (0, 1)$

   $\frac{\partial f}{\partial t} = (y + z)s + (x + z)se^{st} + 2(x + y)t \to 2 \quad$ at $\quad (s, t) = (0, 1)$

5. [10 points]

$f_x = 2xy = 0$ so $x = 0$ or $y = 0$.

$f_y = x^2 + 2y - 1 = 0$. Use $x = 0$ gives $y = 1/2$. Use $y = 0$ gives $x = \pm 1$.

$(0, 1/2) : D = f_{xx} f_{yy} - f_{xy}^2 = 2(2y) - 4x^2 = +2$. Minimum.

$(1, 0) and (-1, 0) : D = -4$. Saddles.

Contour line $f = 0$ gives $y(x^2 + y - 1) = 0$ that is $y = 0$ or $y = 1 - x^2$.