Math 11 Fall 2010 SOLUTIONS/grading.

Important hint: In several, but not all, of the problems below, you can simplify the work by applying one of the theorems from Chapter 17. Think before you calculate! If an integral looks impossible, see if you can use the Divergence Theorem, Stokes' Theorem, etc, to replace it with a simpler integral.

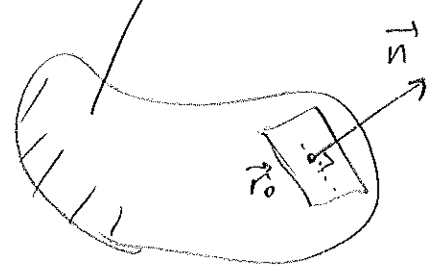
Barnett/Van Erp.

You may also find the following well-known identities useful:

$$\sin^2 x = (1 - \cos 2x)/2 \qquad \cos^2 x = (1 + \cos 2x)/2 \qquad \sin 2x = 2 \sin x \cos x$$

1. [8 points] Find an equation for the tangent plane to the surface $x^2z + 2xy^2 + 3yz^2 = 6$ at the point $(x, y, z) = (1, 1, 1)$.

implicit surface: $F(x,y,z) = \text{const}$



Let $F(x, y, z) := x^2 z + 2xy^2 + 3yz^2$

then $F(x, y, z) = 6$

and $\dfrac{\partial F}{\partial x} = 2xz + 2y^2$

$\dfrac{\partial F}{\partial y} = 4xy + 3z^2$

$\dfrac{\partial F}{\partial z} = x^2 + 6yz$

plug in $(x, y, z) = (1, 1, 1)$

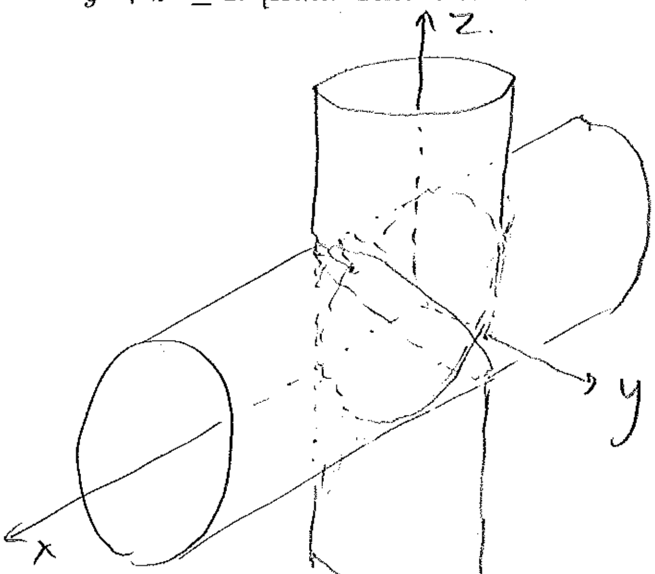
$\left.\dfrac{\partial F}{\partial x}\right|_{(1,1,1)} = 4$, $\left.\dfrac{\partial F}{\partial y}\right|_{(1,1,1)} = 7$. $\left.\dfrac{\partial F}{\partial z}\right|_{(1,1,1)} = 7$

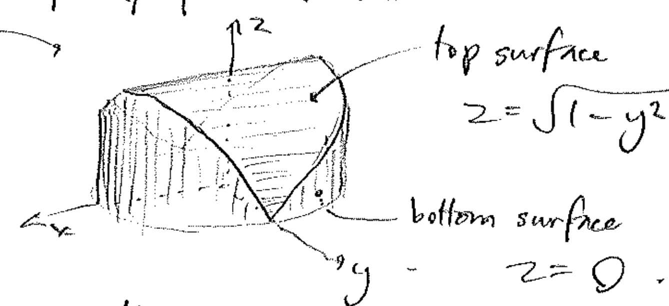
Thus the tangent plane is.

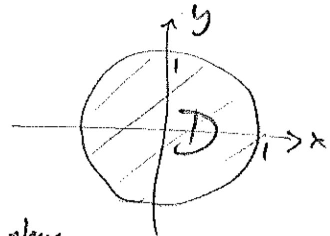
$4(x-1) + 7(y-1) + 7(z-1) = 0$ $\qquad \vec{n} \cdot (\vec{r} - \vec{r_0}) = 0$.

or $4x + 7y + 7z = 18$

2. [10 points] Find the volume of the intersection of the two solid cylinders $x^2 + y^2 \leq 1$ and $y^2 + z^2 \leq 1$. [*Hint:* This is easiest when evaluated in Cartesian coordinates (x, y, z).]







domain of integration ("shadow") in xy plane = unit disc.

volume

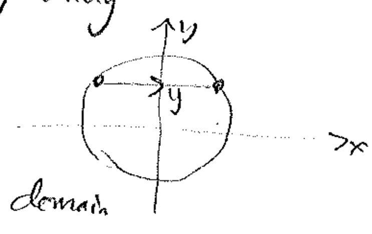
$$V = 2 \iint_D \overbrace{\sqrt{1-y^2}}^{\text{height above each pt in xy plane}} \, dA$$

written as double integral.

hint suggest Cartesians.

do as Type II (x first) since integrating $\sqrt{1-y^2}\,dy$ is nasty.

$$= 2 \underbrace{\int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}}_{x \text{ limits at each } y \text{ for unit disc domain}} \sqrt{1-y^2}\, dx\, dy$$



note $\int \sqrt{1-y^2}\, dx = x\sqrt{1-y^2} + C$

$$= 2 \int_{-1}^{1} \underbrace{2\sqrt{1-y^2}\sqrt{1-y^2}}_{2(1-y^2)}\, dy$$

$$= 2 \left[2y - \frac{2}{3}y^3 \right]_{-1}^{1} = 2\left[4 - \frac{4}{3}\right] = \frac{16}{3}$$

[7 pts if get stuck on doing it as Type I]

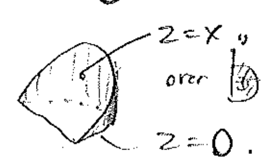
[6 pts for correct triple integral].

[1 pt for $\iiint 1\, dV$]

[−1pt for wrong # congruent pieces].

Alternative way to do:

1/8 of volume



$$\frac{1}{8}V = \int_{-\pi/2}^{\pi/2} \int_0^1 \overbrace{r\cos\theta}^{x}\, r\, dr\, d\theta = 2/3$$

ie 8 pieces.

3. [6 points]

(a) Either find the limit and prove that it is what you claim, or prove that it does not exist:

$$\lim_{(x,y)\to(0,0)} \frac{x}{1+\sqrt{x^2+y^2}} \qquad \lim_{(x,y)\to(0,0)} \frac{x}{1+\sqrt{x^2+y^2}} = 0$$

pf:

① $\frac{x}{1+\sqrt{x^2+y^2}}$ is continuous at $(0,0)$. since quotient of continuous funcs, on domain of defn.

then $\lim_{(x,y)\to(0,0)} \frac{x}{\sqrt{1+x^2+y^2}} = \frac{x}{\sqrt{1+x^2+y^2}}\Big|_{x=0,\ y=0} = 0$

② Let $x = r\cos\theta$ $y = r\sin\theta$. $\Rightarrow \frac{x}{1+\sqrt{x^2+y^2}} = \frac{r\cos\theta}{1+r}$

Let $\delta(r,\theta) := \frac{r\cos\theta}{1+r}$ then $\frac{-r}{1+r} \le \delta(r,\theta) \le \frac{r}{1+r}$

$\lim_{r\to 0} \frac{-r}{1+r} = \lim_{r\to 0} \frac{r}{1+r} = 0 \Rightarrow \lim_{r\to 0} \delta(r,\theta) = 0 \Rightarrow \lim_{(x,y)\to(0,0)} \frac{x}{1+\sqrt{x^2+y^2}} = 0$

or $|\delta(r,\theta)| \le \frac{r}{1+r} \to 0 \ (r\to 0)$.

(b) Either find the limit and prove that it is what you claim, or prove that it does not exist:

$$\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+y^2}$$

The limit does not exist.

pf: Let $y = kx$.

then $\frac{2xy}{x^2+y^2} = \frac{2x\cdot kx}{x^2+k^2x^2} = \frac{2k}{1+k^2}$. (when $x \neq 0$).

So $\lim_{x\to 0;\ y=kx} \frac{2xy}{x^2+y^2} = \lim_{x\to 0} \frac{2k}{1+k^2} = \frac{2k}{1+k^2}$

$\frac{2k}{1+k^2}$ depends on k.

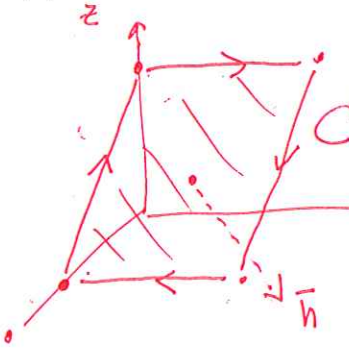
$\therefore$ The limit does not exist.

6

4. [10 points] Let C be the rectangle that consists of the four oriented straight line segments from $(1,0,0)$ to $(0,0,1)$; from $(0,0,1)$ to $(0,1,1)$; from $(0,1,1)$ to $(1,1,0)$; and finally from $(1,1,0)$ back to $(1,0,0)$. Notice that the closed curve C is *oriented* as indicated, for each of its four pieces in the direction from the first to the second point mentioned above.

(a) Find the value of the line integral

$$\oint_C \sin x\,dx + \ln y\,dy + xyz\,dz.$$



$\vec{F} = \langle \sin x, \ln y, xyz \rangle$

Surface S, oriented downward.

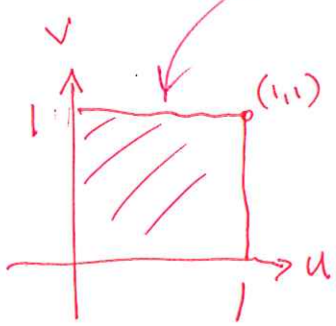
2

$z = f(x,y) = 1 - x$

$\vec{r}(u,v) = \langle u, v, 1-u \rangle$

up *

$\vec{N} = \langle -f_x, -f_y, 1 \rangle = \langle 1, 0, 1 \rangle$



orientation

1 *

$\boxed{-\frac{1}{6}}$

$\operatorname{curl}\vec{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ \sin x & \ln y & xyz \end{vmatrix} = \langle xz, -yz, 0 \rangle$ 1

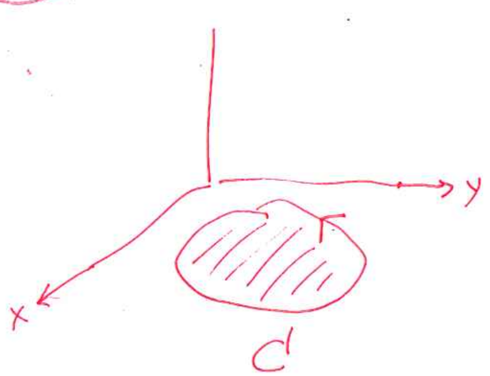
$\bar{N}\cdot \operatorname{curl}\bar{F} = xz = x(1-x) = u(1-u)$ 1

1

$\iint_S \operatorname{curl}\bar{F}\cdot d\bar{S} = \int_0^1\int_0^1 x(1-x)\,dx\,dy = \int_0^1 x(1-x)\,dx = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$ 1

4

(b) Let C_2 be an *arbitrary* circle in the xy-plane (i.e., with $z = 0$), oriented counter-clockwise. Explain why $\oint_{C_2} \sin x\,dx + \ln y\,dy + xyz\,dz = 0$.



$\vec{n} = \langle 0, 0, 1 \rangle$

in fact, $\operatorname{curl}\vec{F} = \vec{0}$ when $z=0$!

$\operatorname{curl}\vec{F}\cdot\vec{n} = 0$

$\Rightarrow \iint_S \bar{F}\cdot\bar{n}\,dS = \oint_C \bar{F}\cdot d\bar{r} = 0.$

S = disk in xy-plane

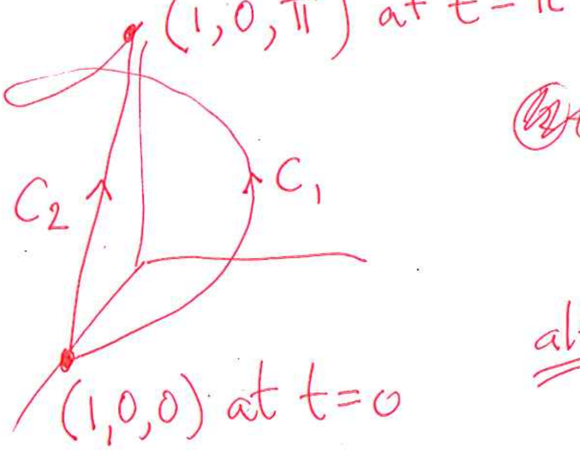
𝍸

this was actually undefined for $x \le 0$ (our mistake), so we gave points for noticing this.

5. [9 points] Let C be the segment of a helix parametrized as $\mathbf{r}(t) = \langle \cos 2t, \sin 2t, t\rangle$, with $0 \le t \le \pi$. The curve C is oriented in the direction from $t = 0$ to $t = \pi$. For the vector field

$$\mathbf{F} = \langle \ln x, e^{y^2}, \sin z\rangle,$$

evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$.



• 2 points if you notice ~~it is conservative~~ curl = 0

• 3 if "conservative"

③ to find C_2

⑤

$$\int_{C_1} F \cdot dr \overset{①}{=} \int_{C_2} F \cdot dr$$

② or STOKES $\boxed{\text{curl} = \vec{0}}$

alt. (1) by Stokes' Thm

(2) because $\bar{F}$ is conservative: $\text{curl}\,\bar{F} = \vec{0}$.

Then $\int_{C_2} \bar{F} \cdot d\bar{r}$: 1. $\vec{r}(t) = (1,0,0) + t(0,0,\pi) = \langle 1, 0, \pi t\rangle$. $0 \le t \le 1$.

④ for this calculation.

SET-UP 2. $\bar{F} = \langle 0, 1, \sin \pi t\rangle$

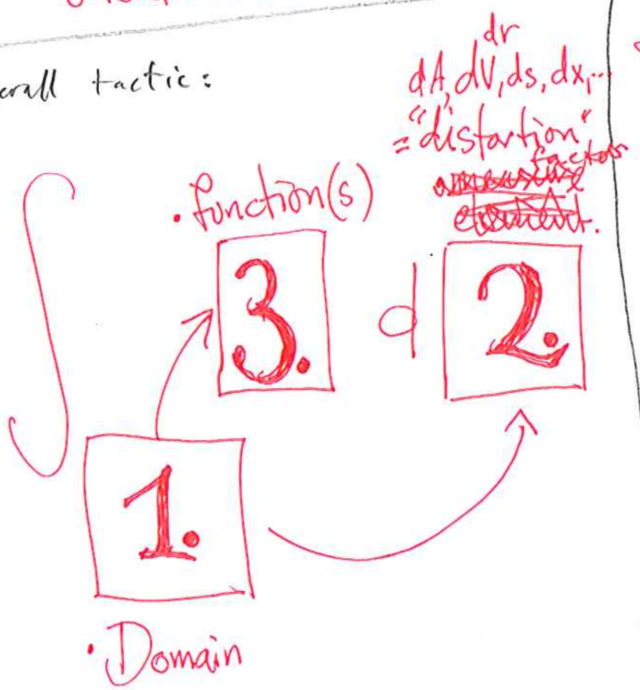
3. $\bar{r}'(t) = \langle 0, 0, \pi\rangle$

(−1) for wrong orientation.

(−1) for calc error

$$\int_0^1 \bar{F} \cdot \bar{r}'\, dt = \int_0^1 \pi \sin \pi t\, dt = \int_0^{\pi} \sin u\, du = 2.$$

Overall tactic:



4 • doing direct $\int_C \bar{F} \cdot d\bar{r}$ with wrong answer / or just stuck.

5 • doing $\int_C \bar{F} \cdot d\bar{r}$, then trying $\text{curl}\,\vec{F} = \vec{0}$ (but nothing from there)

$\int \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$

$\frac{\partial F}{\partial x}(x-x_0) + \frac{\partial F}{\partial y}(y-y_0) + \frac{\partial F}{\partial z}(z-z_0)$

6. [9 points] Let S be the part of the paraboloid $z = x^2 + y^2$ lying under the plane $2y + z = 3$. The surface S is oriented by *downward* pointing normal vectors. Calculate the flux of the vector field

$$\mathbf{F} = \langle 3z - 2x, y - x, z + 2x \rangle$$

across the surface S, i.e., evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$.

$d(P\,dx + Q\,dy)$

$\frac{\partial P}{\partial x} \cancel{dx \wedge dx} + \frac{\partial P}{\partial y} dy \wedge dx$

$+ \frac{\partial Q}{\partial x} dx \wedge dy + \frac{\partial Q}{\partial y} dy \wedge dy$

$\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy$

Divergence Theorem: $\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F} \, dV$ (+1)

Need to close up surface:

$S' = \{(x,y,z) \in \mathbb{R}^3 : z \geq x^2 + y^2,\ 2y + z = 3\}$ (+2)

$\Rightarrow \iint_{S \cup S'} \vec{F} \cdot d\vec{S} = \iiint \operatorname{div} \vec{F} \, dV$

$\operatorname{div} \vec{F} = -2 + 1 + 1 = 0$

$\therefore \iint_{S \cup S'} \vec{F} \cdot d\vec{S} = 0$ (+2)

Integral over S':

x

y

$z = 3 - 2y$

$r_x = \langle 1, 0, 0 \rangle$

$r_y = \langle 0, 1, -2 \rangle$

$r_x \times r_y = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 0 \\ 0 & 1 & -2 \end{vmatrix}$

$= 2\hat{j} + \hat{k}$ (+1)

$= \langle 0, 2, 1 \rangle$

$\vec{F} \cdot \langle 0,2,1 \rangle = 2(y - x) + (z + 2x)$

$= \cancel{2y} - \cancel{2x} + 3 - \cancel{2y} + \cancel{2x}$ (+1)

$= 3$

$\iint_{S'} \vec{F} \cdot d\vec{S} =$

$\iint_C 3 \, dA = 3(4\pi)$ (+2)

$\boxed{12\pi}$

$z = 3 - 2y$

$z = x^2 + y^2$

$3 - 2y = x^2 + y^2$

$x^2 + y^2 + 2y - 3 = 0$

$x^2 + y^2 + 2y + 1 - 4 = 0$

$x^2 + (y+1)^2 = 4$

7. [10 points] Let S be the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies above the plane $z = 1$, and let C denote the curve that is the boundary of S. Finally, let $f(x, y, z) = z^2$.

(a) Evaluate the surface integral $\iint_S f\, dS$.

7

Parametrize

$x = u$

$y = v$

$z = \sqrt{4 - u^2 - y^2}$

$r_u = \langle 1, 0, \tfrac{1}{2}(4-u^2-y^2)^{-1/2}(-2u) \rangle$

$r_v = \langle 0, 1, \tfrac{1}{2}(4-u^2-y^2)^{-1/2}(-2v) \rangle$

$r_u \times r_v = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & \frac{-u}{\sqrt{4-u^2-v^2}} \\ 0 & 1 & \frac{-v}{\sqrt{4-u^2-v^2}} \end{vmatrix}$

+3

$= \frac{u}{\sqrt{4-u^2-v^2}}\,\hat{\imath} + \frac{v}{\sqrt{4-u^2-v^2}}\,\hat{\jmath} + \hat{k}$

$|r_u \times r_v| = \left(\frac{u^2}{4-u^2-v^2} + \frac{v^2}{4-u^2-v^2} + \frac{4-u^2-v^2}{4-u^2-v^2} \right)^{1/2} = \left(\frac{4}{4-u^2-v^2}\right)^{1/2}$

$\iint_S f\, dS = \cancel{\int_{-\sqrt{3}}^{\sqrt{3}} \int_{-\sqrt{3-y^2}}^{\sqrt{3-y^2}} (4-x^2-y^2)^{1/2} \sqrt{\frac{1+x^2+y^2}{4-x^2-y^2}}\, dx\, dx} = \cancel{\iint (4-x^2-y^2)(1+x^2+y^2)\, dx\, dy}$

+2

+1

$= \iint (4-x^2-y^2) \frac{2}{\sqrt{4-x^2-y^2}}\, dA = \iint 2\sqrt{4-x^2-y^2}\, dA = \int_0^{2\pi}\int_0^{\sqrt{3}} 2\sqrt{4-r^2}\, r\, dr\, d\theta$

$u = 4 - r^2 \qquad du = 2r\,dr$

$= \int_0^{2\pi}\int_1^4 \sqrt{u}\, du\, d\theta$

$= 2\pi \left(\tfrac{2}{3} u^{3/2}\right)\Big|_1^4$

$= \frac{4\pi}{3} \cdot (8-1)$

$= \frac{4\pi}{3} \cdot 7$

$= \frac{28\pi}{3}$ +1

3 (b) Evaluate the line integral $\int_C f\, ds$.

+1

C : circle of radius $\sqrt{3}$, and $z \equiv 1$ on C

$\int_C f\, ds = 2\pi(\sqrt{3}) = \boxed{2\sqrt{3}\pi}$ +2

10

PTO for: Alternate solution

(a) Parametrize: $x = 2\sin\phi\cos\theta \quad y = 2\sin\phi\sin\theta \quad z = 2\cos\phi$

$r_\phi = \langle 2\cos\phi\cos\theta,\ 2\cos\phi\sin\theta,\ -2\sin\phi\rangle$

$r_\theta = \langle -2\sin\phi\sin\theta,\ 2\sin\phi\cos\theta,\ 0\rangle$

(+3)

(or if they remember the volume element)

$$r_\phi \times r_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2\cos\phi\cos\theta & 2\cos\phi\sin\theta & -2\sin\phi \\ -2\sin\phi\sin\theta & 2\sin\phi\cos\theta & 0 \end{vmatrix}$$

$$= 4\sin^2\phi\cos\theta\,\hat{i} + 4\sin^2\phi\sin\theta\,\hat{j} + (4\cos^2\theta\sin\phi\cos\phi + 4\sin^2\theta\sin\phi\cos\phi)\,\hat{k}$$

$$|r_\phi \times r_\theta| = \sqrt{16\sin^4\phi\cos^2\theta + 16\sin^4\phi\sin^2\theta + 16\sin^2\phi\cos^2\phi}$$

$$= \sqrt{16\sin^4\phi + 16\sin^2\phi\cos^2\phi}$$

$$= 4\sin\phi\sqrt{\sin^2\phi + \cos^2\phi} = 4\sin\phi$$

(+1) $\left[z^2 = 4\cos^2\phi \right.$ $\quad z = 1 \Rightarrow \cos\phi = \frac{1}{2} \quad \phi = \pi/3$

$$\int_0^{2\pi}\int_0^{\pi/3} (4\cos^2\phi)(4\sin\phi)\, d\phi\, d\theta = 32\pi \int_0^{\pi/3} \cos^2\phi\sin\phi\, d\phi$$

$u = \cos\phi$

$du = -\sin\phi\, d\phi$

(+2)

$$-\int u^2\, du = -\frac{u^3}{3} = -\frac{\cos^3\phi}{3}\Bigg|_0^{\pi/3} = -\frac{\cos^3(\pi/3)}{3} + \frac{\cos^3(0)}{3}$$

$$= \frac{\left(-\frac{1}{8}\right)}{3} + \frac{1}{3}$$

$$= \frac{1}{3} - \frac{1}{3\cdot 8} = \frac{8-1}{3\cdot 8} = \frac{7}{3\cdot 8}$$

$= \cancel{512\pi\left(\frac{21}{24}\right) = 8\pi(21) = \boxed{168\pi}}$

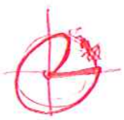
$$32\pi\left(\frac{7}{3\cdot 8}\right) = \frac{28\pi}{3}$$

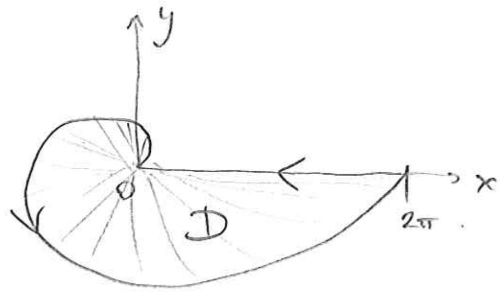
(+1)

8. [9 points] Evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$ where

$$\mathbf{F} = \langle x - y, x + y \rangle$$

and C is the curve given by the polar spiral function $r = \theta$ for $0 \leq \theta \leq 2\pi$ followed by the line segment from $(2\pi, 0)$ to the origin, traversed counter-clockwise. [*Hint:* use Green's Theorem]





$\oint_C F \cdot dr = \cancel{\iint P\,dx \; Q\,dy}$

$(x-y)$

~~for Green's and eval.~~

~~for converting to polar~~

~~for each bound~~

~~for algebra?~~

$$4\left\{ \begin{aligned} &\iint \left(\frac{dQ}{dx} - \frac{dP}{dy}\right) = \\ &\iint 1 - (-1) = \iint 2\, dA \end{aligned} \right.$$

$$\underbrace{\int_0^{2\pi}\int_0^{\theta} 2r\,dr\,d\theta}_{3} = \underbrace{\int_0^{2\pi} r^2\Big|_0^{\theta} d\theta = \int_0^{2\pi} \theta^2 = \frac{8}{3}\pi^3}_{2}$$

if see 2 · area but then use area of Δ, -3

if just calculate 2, don't even set up integral 1/9

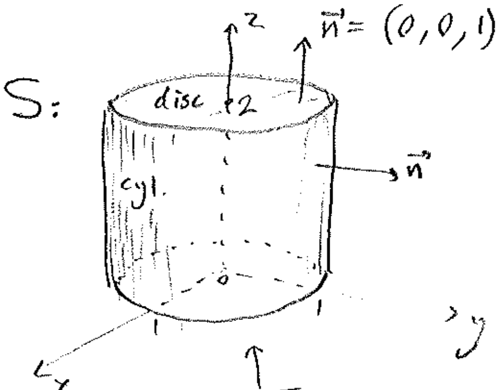
didn't use Green's -5

9. [10 points] Let S be the union of the cylinder $x^2 + y^2 = 1$, $0 \le z \le 2$, with normal pointing outwards, with the disc $x^2 + y^2 \le 1$, $z = 2$, with normal pointing upwards. Let

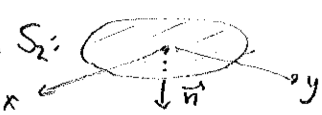
$$\mathbf{F} = \langle x + z^2, y + e^{z^2}, z + x^2 \rangle \quad = \quad \langle P, Q, R \rangle$$

be a vector field defined in $\mathbb{R}^3$. Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$.

S:



S is open underneath; can close with S_2:



Since e^{z^2} is not integrable w.r.t. dz, suggests not doing $\iint_S \vec{F} \cdot d\vec{S}$ directly.

Rather let E = solid cylinder $x^2+y^2 \le 1$, $0 \le z \le 2$.

Div. Thm:

$$\iint_S \vec{F} \cdot d\vec{S} + \iint_{S_2} \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F} \, dV \quad (*)$$

[2 pts]

where S_2 = disc $x^2+y^2 \le 1$, $z=0$, $\vec{n} = (0,0,-1)$

So $S + S_2$ has normals oriented outwards from E, everywhere.

$$\operatorname{div} \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

$$= 1 + 1 + 1 = 3 \quad \text{everywhere.}$$

[2 pts]

So RHS of $(*)$ is $\iiint_E \operatorname{div}\vec{F} \, dV = \iiint_E 3 \, dV = 3$ (cylinder volume)

$$= 3 \cdot \underset{h}{2} \cdot \pi \underset{r=1}{r^2}$$

$$= 6\pi.$$

[3 pts]

And, $\displaystyle\iint_{S_2} \vec{F} \cdot d\vec{S} = \iint_{\text{unit disc}} \left(x+z^2, y+e^{z^2}, z+x^2\right) \cdot (0,0,-1) \, dA$

$$= \iint_{\text{unit disc}} -(\underset{\substack{z=0 \\ \text{on this disc } S_2}}{z} + x^2) \, dA = -\int_0^{2\pi}\int_0^1 r^2 \cos^2\theta \, r \, dr \, d\theta \quad \text{trig. identity.}$$

$$= -\int_0^1 r^3 dr \cdot \int_0^{2\pi} \frac{1}{2}(1 + \cos 2\theta) \, d\theta.$$

$$= -\frac{1}{4} \cdot \pi \qquad \text{since } \int_0^{2\pi} \cos 2\theta \, d\theta = 0$$

[1 pts].

Using $(*)$,

$$\iint_S \vec{F} \cdot d\vec{S} = 6\pi - \left(-\frac{\pi}{4}\right) = \frac{25\pi}{4}.$$

10. [9 points] Let $\mathbf{F}(x, y)$ be the force field $(\frac{1}{y}, 1 - \frac{x}{y^2})$ defined in the upper half plane $y > 0$.

(a) Either find a scalar function f such that $\mathbf{F} = \nabla f$, or else explain why such a function does not exist.

$F = \nabla f = \langle f_x, f_y \rangle$

2 for f_x (or whichever they start with)

$f_x = \frac{1}{y}$

2 for integrating and setting equal

$f = \frac{x}{y} + g(y)$

5

1 for integrating $g'(y)$

$f_y = \frac{-x}{y^2} + g'(y) = 1 - \frac{x}{y^2}$

$g'(y) = 1$

$g(y) = y + K$

-3 if get partials wrong but good argument

-1 if had right answer w/ $\frac{x}{y} + g(y)$ but then solved for $g(y)$ and plugged in to wrong part

$\Rightarrow f = \frac{x}{y} + y + k$

(b) Find the work done by the force field $\mathbf{F}$ in moving a particle along the curve C parametrized by $(1 + t^2, 1 + \sin \pi t)$ starting at $t = 0$ and ending at $t = 1$.

4

fund theorem: $f(b) - f(a) = f(2, 1) - f(1, 1)$

$= 3 - 2 = 1$

remember to use fund theorem: 2

plug in values for t: 1

evaluating: 1

if actually get integral and compute correctly get full points

if set up integral right, maybe 2 pts?

-1 bad algebra / -1 using wrong form

$1 + \frac{1}{2}$

11. [10 points] True or False? On this question (and only on this question) you do not need to show work or explain your answer. (Guessing is allowed and can gain you credit. Therefore, do not skip any items.)

$\vec{\nabla}\cdot\vec{\nabla}f = \nabla^2 f$ = Laplacian,

(a) True / False. For any smooth function $f(x,y,z)$ the divergence of ∇f is zero.

(b) True / False. For any smooth function $f(x,y,z)$ the curl of ∇f is zero. try it & use Clairaut.

(c) True / False. If S is a *closed* oriented surface in $\mathbb{R}^3$, and $\mathbf{F}$ a smooth vector field, then always $\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = 0$. Two proofs: a) div curl $\equiv 0$ & use Div. Thm. b) split $S = S_1 + S_2$ along curve C, use Stokes.

(d) True / False. In the conversion of a triple integral from rectangular coordinates (x, y, z) to spherical coordinates (ρ, θ, ϕ) the volume element is transformed as

$$dx\,dy\,dz = \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi.$$

order irrelevant.

(e) True / False. If C is the circle $x^2+y^2 = r^2$ in $\mathbb{R}^2$ oriented clockwise, then $\int_C 1ds = -2\pi r$.
Scalar line integral doesn't change sign when reversed.

(f) True / False. If D is a region in the plane $\mathbb{R}^2$ and C is the boundary of D, oriented counter-clockwise, then $\int_C y\,dx$ is equal to the area of D. $\vec{F} = (y, 0)$ has $Q_x - P_y = -1$ so wrong sign.

(g) True / False. If $\mathbf{F}$ is a vector field in $\mathbb{R}^3$ for which div $\mathbf{F}$ is not zero, then there cannot exist a vector field $\mathbf{G}$ such that $\nabla\times\mathbf{G} = \mathbf{F}$. if F were $\nabla\times G$ then div F = div curl G $\equiv 0$.

(h) True / False. If the domain of a vector field $\mathbf{F}$ is not all of $\mathbb{R}^2$, then $\mathbf{F}$ cannot be conservative. viz. $\left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right)$.

(i) True / False. $\frac{d}{dt}(|\mathbf{r}|^2) = \mathbf{r}'\cdot\mathbf{r}$ where $\mathbf{r}(t)$ is a position as a function of time.
missing factor 2.

(j) True / False. $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = -(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$ for all vectors $\mathbf{a},\mathbf{b},\mathbf{c}$ in $\mathbb{R}^3$.

swapping dot & cross doesn't change sign.