Practice Problems for Exam I

Math 11 Fall 2007

October 18, 2007

These are some practice problems for the first exam. This is just a collection of problems of the sort we might put on exams; we have made no attempt to recreate the balance of topics found in the exam. Anything we covered (through section 15.6) may appear on the exam, whether or not it appears here.

Some of these problems are problems we considered putting on this exam but didn't. (That might be because we wanted other problems, or because there is a very similar problem we did put on the exam.) Some of them were taken from former Math 12 exams. Some of them come from other sources.

If this were a real exam, we would require that you show your work on all but short answer, true/false, and multiple choice problems; your grade would depend on the work you showed and not just on your answer. On short answer, true/false, and multiple choice problems, you would not have to show your work, and we would not give partial credit. You would not be allowed to use your textbook, notes, or calculator.

Also, if this were a real exam, it would be shorter. Much shorter.

1. TRUE or FALSE: There is a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\frac{\partial f}{\partial x} = y \quad \text{and} \quad \frac{\partial f}{\partial y} = x^2.$$

2. TRUE or FALSE: There is a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\frac{\partial f}{\partial x} = x \quad \text{and} \quad \frac{\partial f}{\partial y} = y^2.$$

3. Find the directional derivative of the function

$$f(x, y, z) = 3xy + z^2$$

at the point $(1,\, -2,\, 2)$ in the direction from that point toward the origin.

4. A skier is on a mountain with equation

$$z = 100 - 0.4x^2 - 0.3y^2,$$

where z denotes height.

   (a) The skier is located at the point with xy-coordinates $(1,\, 1)$, and wants to ski downhill along the steepest possible path. In which direction (indicated by a vector $(a,\, b)$ in the xy-plane) should the skier begin skiing?

   (b) The skier begins skiing in the direction given by the xy-vector (a, b) you found in part (a), so the skier heads in a direction in space given by the vector $(a,\, b,\, c)$. Find the value of c.

   (c) A hiker located at the same point on the mountain decides to begin hiking downhill in a direction given by a vector in the xy-plane that makes an angle θ with the vector $(a,\, b)$ you found in part (a). How big should θ be if the hiker wants to head downhill along a path whose slope is at most 0.5 (in absolute value)?

5. Suppose that $f : \mathbb{R}^3 \to \mathbb{R}$ is a differentiable function, $\mathbf{u}$ is a unit vector in $\mathbb{R}^3$, and $\mathbf{r} : \mathbb{R} \to \mathbb{R}^3$ is a differentiable function representing the position of a moving object as a function of time. Let $g(t)$ be the value of f at the object's position at time t. Show that if at time t_0 the object is at position (x_0, y_0, z_0) moving in the direction $\mathbf{u}$ with a speed of 1, then $g'(t_0) = D_{\mathbf{u}}(x_0,\, y_0,\, z_0)$.

   This problem calls for a mathematical argument, not a proof via intuitive physical reasoning.

6. Let $f(x, y, z) = x^2 - y^2 + xyz$ and $\mathbf{v} = \langle 3,\, 4,\, 12\rangle$.

   (a) Find the directional derivative of f in the direction of the vector $\mathbf{v}$ at the point $(1,\, 2,\, -1)$.

(b) Let $\mathbf{r}(t)$ be a differentiable function giving the position of a moving object as a function of time, such that at time $t = 0$ the object is at the point $(1, 2, -1)$ moving in the direction of $\mathbf{v}$ at a speed of 1. Compute $\mathbf{r}'(0)$.

(c) Consider the same moving object whose position function is given in part (b). Let $g(t)$ be the value of f at the object's position at time t. Find $g'(0)$.

7. Sometimes a surface in $\mathbb{R}^3$ is easiest to picture by expressing z as a function of polar coordinates r and θ. We may still want to find the partial derivatives of z with respect to x and y, for example, to draw the gradient field.

   Give expressions for $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ in terms of $\dfrac{\partial z}{\partial r}$, $\dfrac{\partial z}{\partial \theta}$, r and θ. Your expressions should be valid when $x > 0$.

   Recall that when $x > 0$ rectangular (Euclidean) and polar coordinates are related by the formulas

$$x = r\cos(\theta) \qquad y = r\sin(\theta)$$

$$r = \sqrt{x^2 + y^2} \qquad \theta = \arctan\left(\frac{y}{x}\right),$$

   where arctan is the inverse tangent function.

8. Find an equation for the tangent plane to the surface with equation

$$x^2 - y^2 + z^2 = 4$$

   at the point $(2, 1, -1)$.

9. Find parametric equations for the line through the points

$$A = (1, 2, 3) \quad \text{and} \quad B = (2, 1, -1).$$

10. Find an equation in the form $Ax + By + C = D$ for the plane containing the line

$$\frac{x-1}{2} = y + 1 = \frac{z-2}{3}$$

   and the point $C = (2, 0, 3)$.

11. Consider the lines L_1 and L_2 with vector equations

$$\langle x,\, y,\, z\rangle = \langle 1,\, 2,\, 3\rangle + t\,\langle a,\, 1,\, 0\rangle \quad \text{and} \quad \langle x,\, y,\, z\rangle = \langle 2,\, 0,\, 1\rangle + s\,\langle 1,\, 1,\, 0\rangle$$

respectively. Is it possible to choose the constant a so that the lines intersect? (This is not simply a "YES or NO" question. You must explain how you arrived at your conclusion.)

12. Suppose that $\mathbf{u}\times\mathbf{v} = \langle 5,\, 1,\, 1\rangle$, that $\mathbf{u}\cdot\mathbf{u} = 4$, and that $\mathbf{v}\cdot\mathbf{v} = 9$. Then $|\mathbf{u}\cdot\mathbf{v}|$ is equal to:

    (a) 2
    (b) 3
    (c) 4
    (d) 5
    (e) None of these.

13. Find the distance between the planes $x + y + z = 1$ and $x + y + z = 4$.

    (a) 2
    (b) $\sqrt{2}$
    (c) $\sqrt{3}$
    (d) 3
    (e) None of these.

14. Find the tangent plane to $f(x, y) = x^2 + 2y^2$ at the point $(2,\, 1,\, 6)$.

    (a) $x + y + z = 9$
    (b) $4x - 4y - z = -9$
    (c) $4x + 4y - z = 6$
    (d) $4x + 4y + z - 18$
    (e) None of these.

15. TRUE or FALSE: The function f is continuous at the point $(0, 0)$, where

$$f(x, y) = \begin{cases} \dfrac{x^4 + y^4}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0); \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

16. Suppose that $\mathbf{a}$ and $\mathbf{b}$ are two vectors in $\mathbb{R}^3$ such that if $\mathbf{a}$ and $\mathbf{b}$ are drawn emanating from the origin they both lie in the xy-plane, $\mathbf{a}$ in the third quadrant ($x < 0$ and $y < 0$) and $\mathbf{b}$ in the second quadrant ($x < 0$ and $y > 0$.)

    Suppose also that we know $|\mathbf{a}| = 1$ and $|\mathbf{b}| = 2$ and $\mathbf{a} \cdot \mathbf{b} = 1$.

    (a) Is the projection of $\mathbf{b}$ onto $\mathbf{a}$ longer than $\mathbf{a}$, shorter than $\mathbf{a}$, or the same length as $\mathbf{a}$?

    (b) In what direction does $\mathbf{a} \times \mathbf{b}$ point?

    (c) Find the length of $\mathbf{a} \times \mathbf{b}$.

    Be sure to explain your answers.

17. A point moves along the intersection of the surface

$$z = x^2 + y^2$$

with the plane

$$x + y = 2$$

from the point $(2, 0, 4)$ to the point $(0, 2, 4)$, in such a way that $x = 2 - t$ (where t denotes time and x denotes the x-coordinate of the point's position at time t).

Find the

(a) velocity

(b) acceleration

(c) unit tangent vector $\mathbf{T}$

(d) speed

when the point is at position $(1, 1, 2)$.

18. Approximate $\displaystyle\int_{-0.1}^{0.1} e^{-t^2}\,dt$ using a linear approximation to the function $g(x,y) = \displaystyle\int_x^y e^{-t^2}\,dt$.

19. Suppose a point moves along the surface $z = f(x,y)$ with its position at time t given by $\vec{r}(t) = (x(t),\, y(t),\, z(t))$. Notice that this means

$$z(t) = f(x(t),\, y(t)).$$

At time t_0 the point is at position $(x_0,\, y_0,\, z_0) = (x_0,\, y_0,\, f(x_0, y_0))$.

(a) Write down an expression for a vector that is normal to the surface $z = f(x,y)$ at the point $(x_0,\, y_0,\, z_0)$. Your expression will involve the partial derivatives of f at $(x_0,\, y_0)$.

(b) Use the fact that the velocity vector $(x'(t_0),\, y'(t_0),\, z'(t_0))$ is tangent to the surface, and therefore normal to the vector you found in part (a), to solve for $z'(t_0)$ in terms of $x'(t_0)$, $y'(t_0)$, and the partial derivatives of f at $(x_0,\, y_0)$.

(c) Now use the chain rule to compute $z'(t_0)$. Your answer should be in terms of $x'(t_0)$, $y'(t_0)$, and the partial derivatives of f at $(x_0,\, y_0)$.
In fact, your answer should be the same as your answer to part (b). You can view parts (a) and (b) as a proof of the chain rule in this case.

20. (Short answer problem.)

(a) What is the area of the triangle with corners $(0,\, 0,\, 0)$, $(0,\, 1,\, -1)$ and $(1,\, 0,\, 1)$?

(b) Give an equation for the plane containing $(0,\, 0,\, 0)$ and parallel to the plane with equation $3x + 2y - z = 8$.

(c) True or False? If $f : \mathbb{R}^2 \to \mathbb{R}$ is a function with continuous second partial derivatives, then $f_{xy} = f_{yx}$.

21. A spaceship moves so that its position at time t, for $0 \le t \le 1$, is $\left(t,\, t,\, t^{\frac{3}{2}}\right)$. At time $t = 1$ the engines are turned off, so that the spaceship continues to move at the same velocity it had reached at $t = 1$.

(a) Find the arc length of the path traveled by the spaceship between times $t = 0$ and $t = 1$.

(b) Where is the spaceship at time $t = 2$?

22. S is the surface with equation $z = x^2 + 2xy + 2y$.

    (a) Find an equation for the tangent plane to S at the point (1, 2, 9).

    (b) At what points on S, if any, does S have a horizontal tangent plane?

23. (a) Show that if $\mathbf{v}$ is any vector function of t and $|\mathbf{v}|$ is constant, then $\mathbf{v}$ is normal (orthogonal, or perpendicular) to $\dfrac{d\mathbf{v}}{dt}$.

    Hint: Express $|\mathbf{v}|$ using the dot product, and remember that we have a "dot product rule" for differentiation.

    (b) Use the result of part (a) to show that if an object travels with constant speed, then its acceleration is normal to its direction of motion.

    This agrees with our physical intuition. Acceleration in the direction of motion should correspond to changing speed, and acceleration normal to the direction of motion should correspond to changing direction.

24. (Short answer problem.) Match each of the functions below with the correct pictures of its graph and its level curves. There are pictures on the following pages.

    (a) $f(x, y) = xy$. Graph: _________. Level Curves: _________.

    (b) $f(x, y) = y - x^2$. Graph: _________. Level Curves: _________.

    (c) $f(x, y) = \dfrac{1}{x^2 + y^2 + 1}$. Graph: _________. Level Curves: _________.

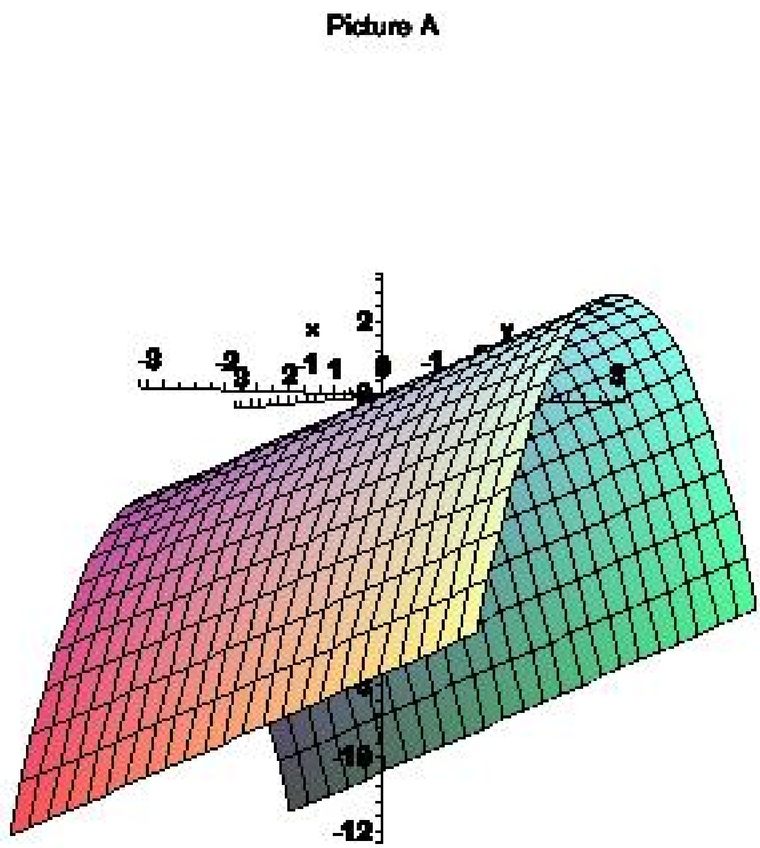

Figure 1: A Picture for Problem 24

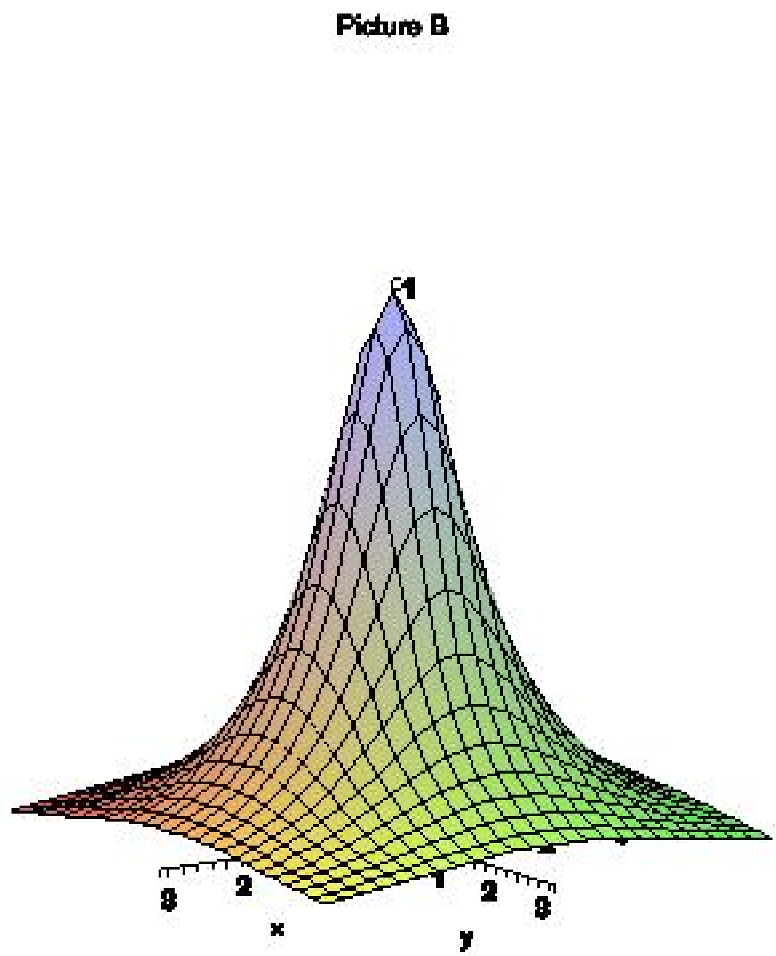

Figure 2: A Picture for Problem 24

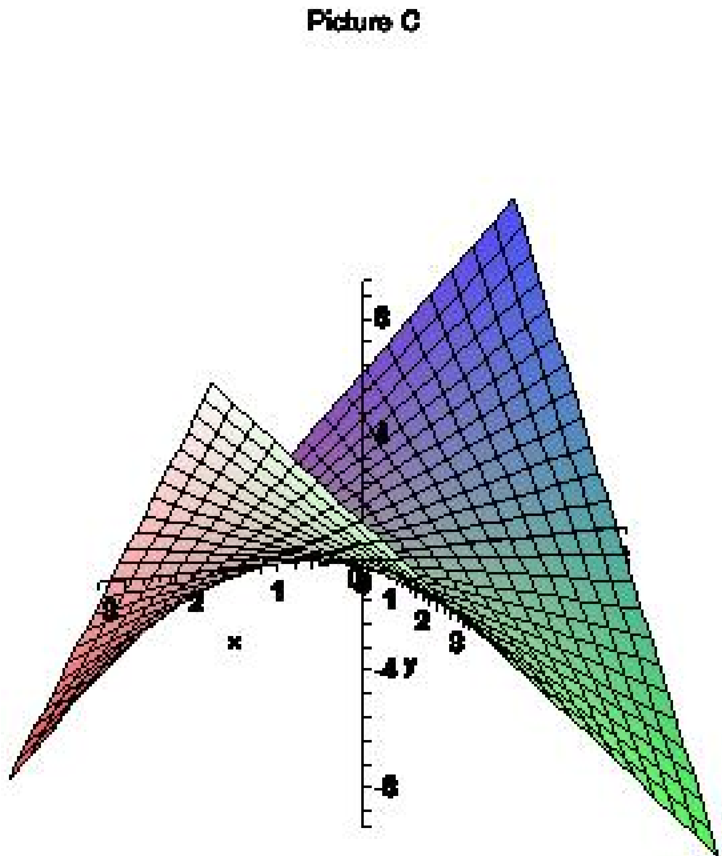

Figure 3: A Picture for Problem 24

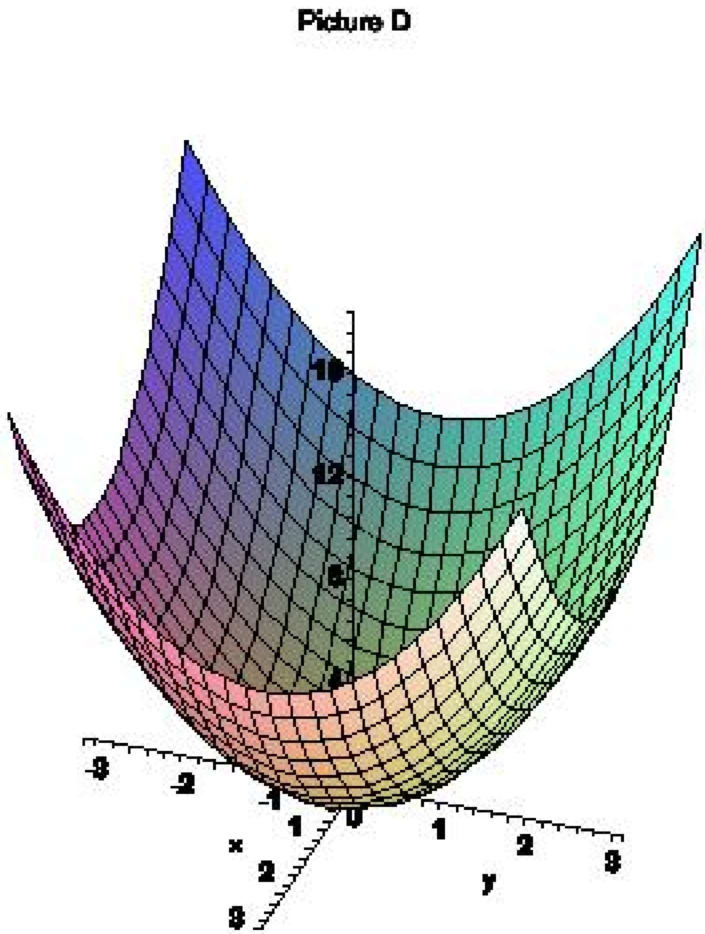

Figure 4: A Picture for Problem 24

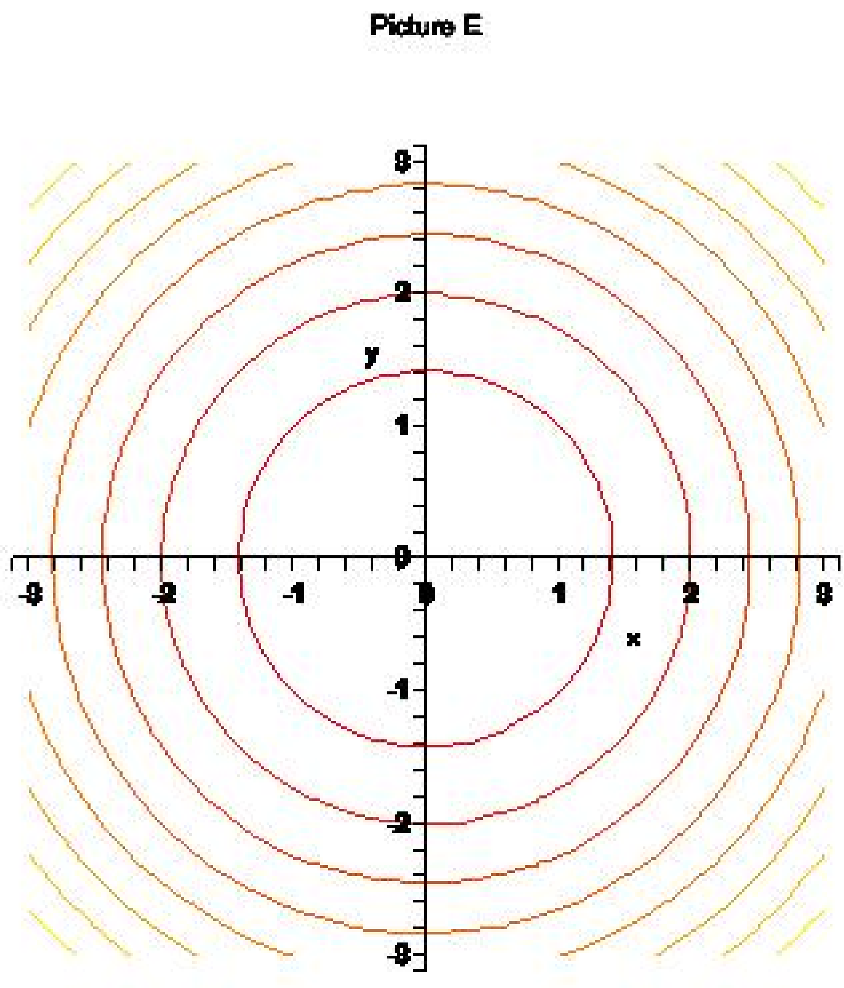

Figure 5: A Picture for Problem 24

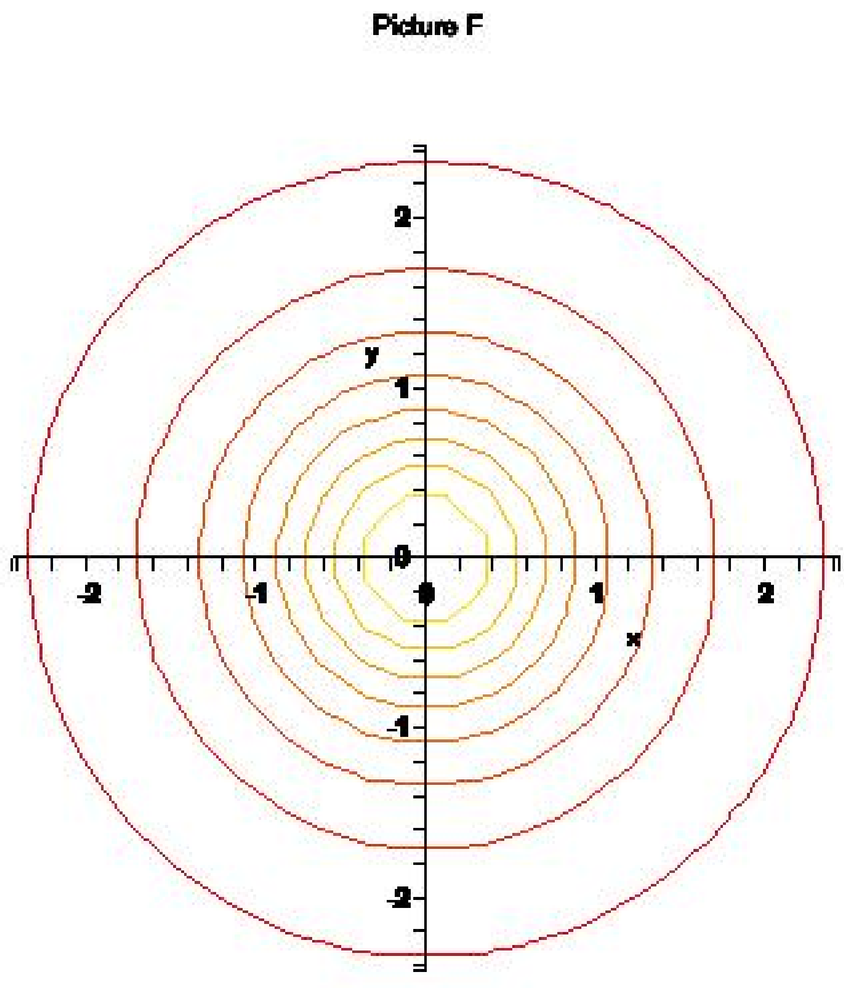

Figure 6: A Picture for Problem 24

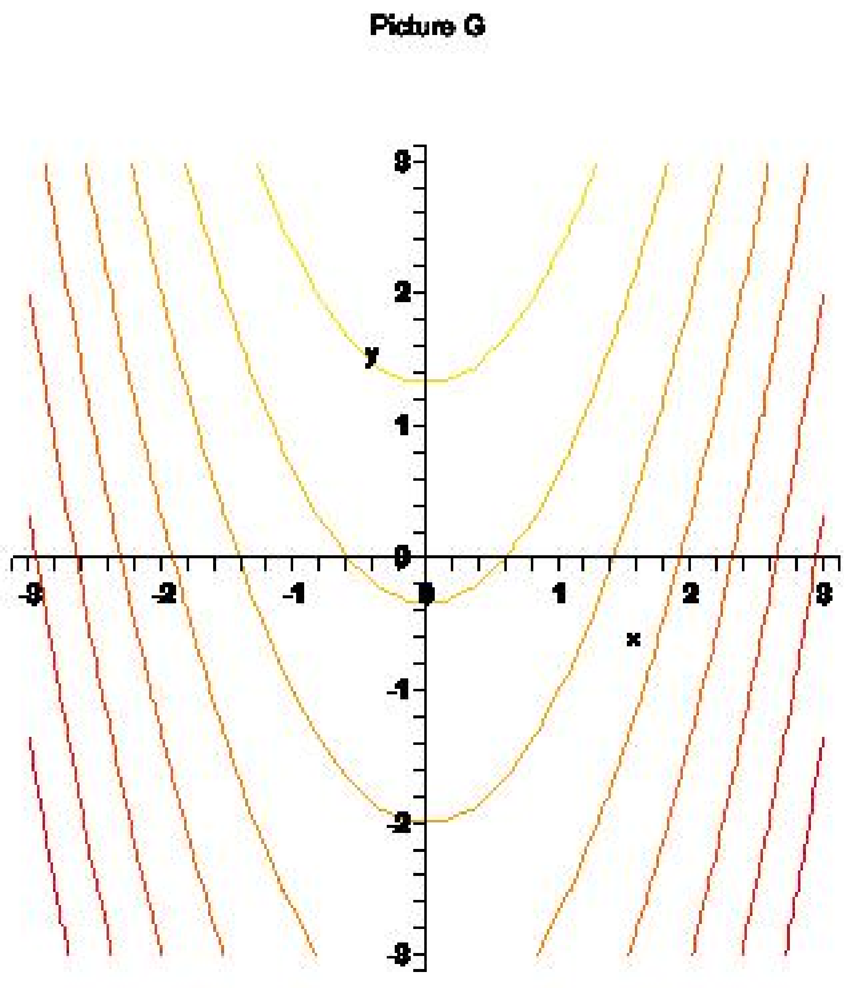

Figure 7: A Picture for Problem 24

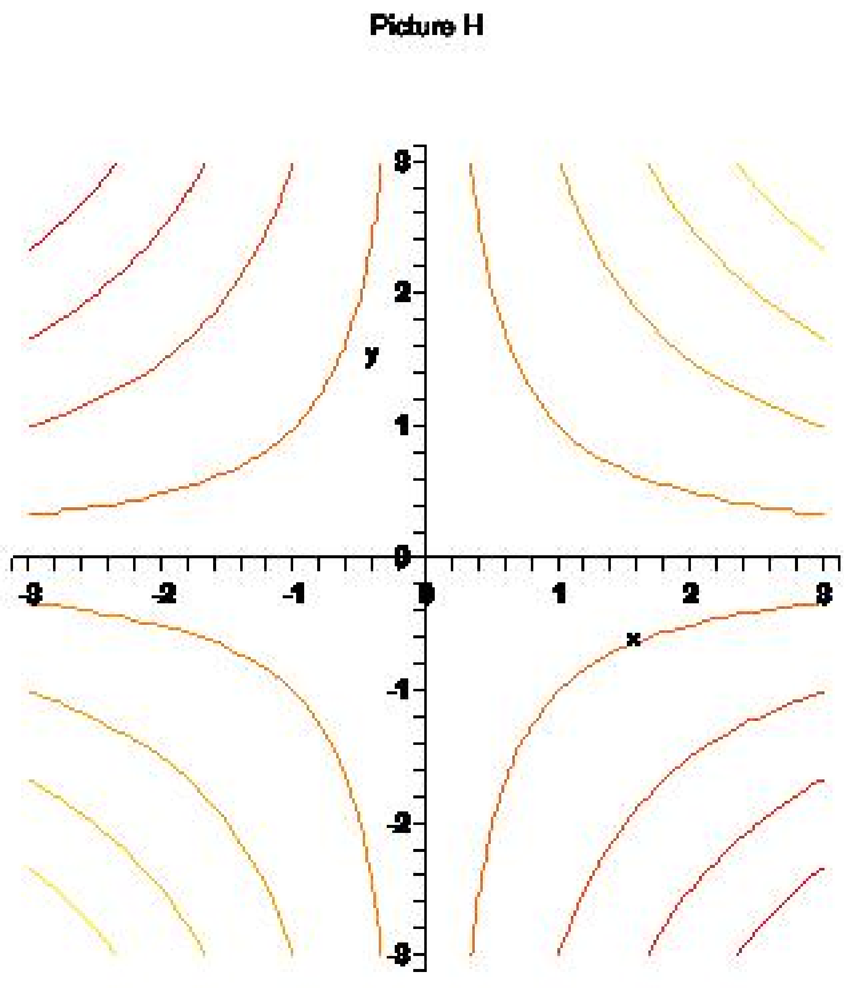

Figure 8: A Picture for Problem 24