1. (15) Let $\mathscr{L}_1$ be the line described by

$$\langle x, y, z\rangle = \langle 1, 0, 1\rangle + t\langle 2, 3, 5\rangle \quad \text{for } t \in \mathbf{R}.$$

Also let $\mathscr{L}_2$ be the line through the points $(2, 1, 1)$ and $(3, 4, 2)$, and let $\mathscr{P}$ be the plane containing the line $\mathscr{L}_2$ and parallel to $\mathscr{L}_1$ (so that $\mathscr{P}$ and $\mathscr{L}_1$ do not intersect).

(a) Find symmetric equations for $\mathscr{L}_1$.

ANS: Since $\mathscr{L}_1$ is the line through $(1, 0, 1)$ are parallel to $\mathbf{u} = \langle 2, 3, 5\rangle$, the symmetric equations are $\boxed{\frac{x-1}{2} = \frac{y}{3} = \frac{z-1}{5}}$

ANS: ______________________

(b) Find parametric equations for $\mathscr{L}_2$.

ANS: $\mathscr{L}_2$ is parallel to the vector $\mathbf{v} = \langle 3-1, 4-1, 2-2\rangle = \langle 1, 3, 1\rangle$. Since it also passes through the point $(2, 1, 1)$ we get parametric equations $\boxed{x = 2 + t,\ y = 1 + 3t \text{ and } z = 1 + t}$.

$x =$ __________ $y =$ __________ $z =$ __________

(c) Find an equation of the form $Ax + By + Cz = D$ for the plane $\mathscr{P}$.

ANS: The plane $\mathscr{P}$ must contain the vectors $\mathbf{u} =< 2, 3, 5 >$ (because it is parallel to $\mathscr{L}_1$) and $\mathbf{v} =< 1, 3, 1 >$ (because it contains the line $\mathscr{L}_2$). Hence the cross product $\mathbf{u}\times\mathbf{v} = \langle -12, 3, 3\rangle$ is normal to the plane. Hence $\mathscr{P}$ is the plane through $(2, 1, 1)$ with normal vector $\mathbf{n} = -\frac{1}{4}\langle -12, 3, 3\rangle = \langle 4, -1, 1\rangle$. Thus $\mathscr{P}$ is given by

$$4(x-2) - (y-1) - (z-1) = 0,$$

or $\boxed{4x - y - z = 6}$.

ANS: ______________________

2. (10) A moving object has position

$$\mathbf{r}(t) = \langle \sqrt{2}\, t^2, \sqrt{2}\, t^2, t^3\rangle$$

at time t.

(a) At time $t = 1$ find expressions for

(i) The object's position.

ANS: The position is the point whose position vector is $\mathbf{r}(1)$. Thus the answer is $\boxed{(\sqrt{2}, \sqrt{2}, 1)}$.

Position = ____________________

(ii) The object's velocity.

ANS: Since $\mathbf{v}(t) = \mathbf{r}'(t) = \langle 2\sqrt{2}t, 2\sqrt{2}t, 3t^2 \rangle$, the velocity is $\mathbf{v}(1) = \boxed{\langle 2\sqrt{2}, 2\sqrt{2}, 3 \rangle}$.

Velocity = ____________________

(iii) The object's speed.

ANS: The speed is $s(1) = |v(1)| = \sqrt{8+8+9} = \boxed{5}$.

Speed = ____________________

(iv) The object's acceleration.

ANS: Since $\mathbf{a}(t) = \mathbf{v}'(t) = \langle 2\sqrt{2}, 2\sqrt{2}, 6t \rangle$, $\mathbf{a}(1)$ is given by $\boxed{\langle 2\sqrt{2}, 2\sqrt{2}, 6 \rangle}$.

Acceleration = ____________________

(b) Find the arc length of the path traveled by the object between times $t = 0$ and $t = 1$.

ANS: We have

$$\begin{aligned} L &= \int_0^1 |\mathbf{r}'(t)|\, dt \\ &= \int_0^1 t\sqrt{16 + 9t^2}\, dt \end{aligned}$$

which, after substituting $u = 16 + 9t^2$ and $du = 18t\, dt$, is

$$\begin{aligned} &= \frac{1}{18} \int_{16}^{25} \sqrt{u}\, du \\ &= \frac{1}{27} u^{3/2} \Big|_{u=16}^{u=25} \\ &= \frac{1}{27}(5^3 - 4^3) \\ &= \boxed{\tfrac{61}{27}}. \end{aligned}$$

Length = ____________________

3. (10) The temperature at the point (x, y, z) in a region of space is given by a function $T(x, y, z)$ such that

$$\frac{\partial T}{\partial x} = 2x + 3y \quad \frac{\partial T}{\partial y} = 3x - 4y + z \quad \frac{\partial T}{\partial z} = y - z^2.$$

If an object moves through that space so that its velocity at the instant it passes through the point $(1, -1, 1)$ is $\mathbf{v} = \langle 2, 0, -3 \rangle$, then how fast is the object's temperature changing at that instant? (You can assume that the units in this problem are seconds, meters and degrees Celsius.)

ANS: Here $T = T(x, y, z)$ and each of x, y and z are functions of t. At the instant in question, the derivatives of x, y and z are given by the vector $\mathbf{v}$. Thus

$$\frac{dx}{dt} = 2 \quad \frac{dy}{dt} = 0 \quad \text{and} \quad \frac{dz}{dt} = -3.$$

On the other hand, at that instant, the partials of T determined by $(x, y, z) = (1, -1, 1)$. Hence

$$\frac{\partial T}{\partial x} = -1 \quad \frac{\partial T}{\partial y} = 8 \quad \text{and} \quad \frac{\partial T}{\partial z} = -2.$$

Now we use the chain rule to compute that at the instant in question

$$\begin{aligned}\frac{dT}{dt} &= \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} + \frac{\partial T}{\partial z}\frac{dz}{dt} \\ &= (-1)(2) + (8)(0) + (-2)(-3) \\ &= \boxed{4}.\end{aligned}$$

ANS: ______________________________

4. (15) Arthur Dent and Ford Prefect[1] inexplicably find themselves located in a region of Vogon space where the temperature at coordinates (x, y, z) is given by

$$T(x, y, z) = x^2 - xz + yz^2 + x.$$

Improbably, they are located at the point $(1, -1, 2)$.

[1] Arthur Dent, Ford Prefect and Vogons are characters from a "five book trilogy" (sic) written by Douglas Adams beginning with *The Hitchhikers Guide to the Galaxy*, in which they discover that the answer to the ultimate question about life, the universe and everything is, well, 42.

(a) If they head directly towards earth (located at the origin), will the surrounding temperature increase or decrease?

ANS: We want the sign of the directional derivative of T in the direction of $-\langle 1,-1,2\rangle$. So let $\mathbf{u} = \frac{1}{\sqrt{6}}\langle -1,1,-2\rangle$. Since $\nabla T(x,y,z) = \langle 2x - z + 1, z^2, -x + 2yz\rangle$ and $\nabla T(1,-1,2) = \langle 1,4,-5\rangle$, $D_{\mathbf{u}}T(1,-1,2) = \frac{1}{\sqrt{6}}\langle 1,4,-5\rangle \bullet \langle -1,1,-2\rangle = \frac{13}{\sqrt{6}}$.

Since this value is positive, the temperature will $\boxed{\text{increase}}$.

____

(b) In what direction should they head so that the surrounding temperature increases as fast as possible? (Your answer should be a unit vector.)

ANS: Just a unit vector in the direction of $\nabla T(1,-1,2)$. So $\boxed{\mathbf{u} = \frac{1}{\sqrt{42}}\langle 1,4,-5\rangle}$.

$\mathbf{u} =$ ______________________________

(c) What is the maximum rate of temperature increase (i.e., the directional derivative in that direction)?

ANS: The length of $\nabla T(1,4,-5)$, or $\boxed{\sqrt{42}}$. (In memory of Douglas Adams, the answer really should have been just "42", but how many of you would have recognized that $\sqrt{1764} = 42$?)

ANS: ______________________________

5. (10) Suppose that $\mathbf{a}$ and $\mathbf{b}$ are nonzero vectors.

(a) Under what conditions on $\mathbf{a}$ and $\mathbf{b}$ are the scalar projections $\operatorname{comp}_{\mathbf{a}}\mathbf{b}$ and $\operatorname{comp}_{\mathbf{b}}\mathbf{a}$ equal? Justify your assertions.

ANS: Recall that if θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, then $\operatorname{comp}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a}\bullet\mathbf{b}}{|\mathbf{a}|} = |\mathbf{b}|\cos(\theta)$. Thus if $\operatorname{comp}_{\mathbf{a}}\mathbf{b} = \operatorname{comp}_{\mathbf{b}}\mathbf{a}$, then

$$|\mathbf{b}|\cos(\theta) = |\mathbf{a}|\cos(-\theta) = |\mathbf{a}|\cos(\theta).$$

It follows the either $\cos(\theta) = 0$ or $|\mathbf{a}| = |\mathbf{b}|$. Therefore the scalar projections are equal if and only if $\boxed{\text{either } \mathbf{a} \text{ and } \mathbf{b} \text{ are orthogonal or } \mathbf{a} \text{ and } \mathbf{b} \text{ have the same length}}$.

(b) Under what conditions on $\mathbf{a}$ and $\mathbf{b}$ are the vector projections $\operatorname{proj}_{\mathbf{a}}\mathbf{b}$ and $\operatorname{proj}_{\mathbf{b}}\mathbf{a}$ equal? Justify your assertions.

ANS: Recall that $\operatorname{proj}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a}\bullet\mathbf{b}}{|\mathbf{a}|^2}\mathbf{a}$, and that $|\operatorname{proj}_{\mathbf{a}}\mathbf{b}| = \operatorname{comp}_{\mathbf{a}}\mathbf{b}$. Thus if $\operatorname{proj}_{\mathbf{a}}\mathbf{b} = \operatorname{proj}_{\mathbf{b}}\mathbf{a}$, then the scalar projections are equal, and we already know that either $\mathbf{a}\bullet\mathbf{b} = 0$ or that $|\mathbf{a}| = |\mathbf{b}|$. But if $\mathbf{a}$ and $\mathbf{b}$ are orthogonal then both vector projections are the zero vector. On the other hand, if $|\mathbf{a}| = |\mathbf{b}|$ and $\operatorname{proj}_{\mathbf{a}}\mathbf{b} = \operatorname{proj}_{\mathbf{b}}\mathbf{a}$, then

$$\frac{\mathbf{a}\bullet\mathbf{b}}{|\mathbf{a}|^2}\mathbf{a} = \frac{\mathbf{b}\bullet\mathbf{a}}{|\mathbf{b}|^2}\mathbf{b}$$

implies that $\mathbf{a} = \mathbf{b}$. Thus the vector projections are equal exactly when $\boxed{\mathbf{a} \text{ and } \mathbf{b} \text{ are either orthogonal or equal}}$.

6. (16) Determine whether the following statements are true or false and circle the correct letter. You do not have to justify your answer, and there will be no partial credit awarded on this problem.

T **F** (a) If $\mathbf{v}$ and $\mathbf{u}$ are nonzero vectors, then $|\mathbf{v}\times\mathbf{u}| = |\mathbf{u}\times\mathbf{v}|$.

ANS: **True**: the length of the cross product is the area of the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$.

T **F** (b) The cross product of two unit vectors is always another unit vector.

ANS: **False**: for example, the vectors could be equal. Then the cross product is the zero vector.

T **F** (c) If $\mathbf{u}$ and $\mathbf{v}$ are any vectors such that $\mathbf{u}\bullet\mathbf{v} = 0$ and $\mathbf{v}\times\mathbf{u} = \mathbf{0}$, then either $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$.

ANS: **True**: If θ is the angle between the vectors, then we have $|\mathbf{u}||\mathbf{v}|\cos(\theta) = 0 = |\mathbf{u}||\mathbf{v}|\sin(\theta)$. If $|\mathbf{u}|$ and $|\mathbf{v}|$ were nonzero, this would force both $\sin(\theta)$ and $\cos(\theta)$ to be zero. This is impossible.

T **F** (d) We have $\displaystyle\lim_{(x,y)\to(0,0)} \frac{(x+y)^2}{x^2+y^2} = 0$.

ANS: **False**: Just consider the limits along the coordinate axes.

T **F** (e) Suppose that $\mathbf{r}(t)$ is the position vector of a particle at time t and that $\mathbf{v}(t)$ is the velocity of the particle at time t. If $\mathbf{a}$ is a constant vector such that $\mathbf{r}(t)$ and $\mathbf{a}$ are always orthogonal, then $\mathbf{v}(t)$ and $\mathbf{a}$ are always orthogonal.

ANS: **True**: Let $h(t) = \mathbf{r}(t)\cdot\mathbf{a}$. By assumption $h(t) = 0$ for all t. Therefore $0 = h'(t) = \mathbf{r}'(t)\cdot\mathbf{a} + \mathbf{0}$.

T **F** (f) We have $\lim_{(x,y)\to(0,0)} \dfrac{x^4 - y^4}{x^2 + y^2} = 0$.

ANS: **True**: the expression simplifies to $\lim_{(x,y)\to(0,0)}(x^2 - y^2)$ which is clearly zero.

T **F** (g) There is a function $z = g(x, y)$, with continuous first and second partial derivatives, such that $g_x(x, y) = 3x = g_y(x, y)$ for all x and y.

ANS: **False**: The mixed partials would not be zero — violating Clairaut's Theorem.

T **F** (h) Suppose that $z = f(x, y)$ is a differentiable function and $(x_0, y_0) \in \mathbf{R}^2$. Then there must be a unit vector $\mathbf{u}$ such that $D_{\mathbf{u}}f(x_0, y_0) = 0$.

ANS: **True**: Pick $\mathbf{u}$ orthogonal to $\nabla f(x_0, y_0)$; if $\nabla f(x_0, y_0) = \langle a, b\rangle$, then let $\mathbf{u}$ be a unit vector in the direction of $\langle b, -a\rangle$. (Of course, if $\langle a, b\rangle = \langle 0, 0\rangle$, then any $\mathbf{u}$ will do.)

7. (24) MULTIPLE CHOICE. Circle the best response. No partial credit will be given on this problem and you do not need to justify your answers.

(a) Suppose that $\mathbf{u} \times \mathbf{v} = \langle 5, 1, 1\rangle$, that $\mathbf{u} \bullet \mathbf{u} = 4$ and that $\mathbf{v} \bullet \mathbf{v} = 9$. Then $|\mathbf{u} \bullet \mathbf{v}|$ is equal to

A. 2 **B**. 3 **C**. 4 **D**. 5 **E**. None of These

ANS: If θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, then $|\mathbf{u}\times\mathbf{v}| = \sqrt{27} = |\mathbf{u}||\mathbf{v}|\sin(\theta)$. Therefore $\sin(\theta) = \frac{\sqrt{27}}{6}$. Then $|\cos(\theta)| = \frac{1}{2}$. But $|\mathbf{u}\bullet\mathbf{v}| = |\mathbf{u}||\mathbf{v}||\cos(\theta)| = 2(3)(\frac{1}{2}) = 3$. Therefore the correct answer is $\boxed{\mathbf{B}}$.

(b) If a is a real number and if the line $\dfrac{x-1}{2} = \dfrac{y-2}{3} = \dfrac{z-3}{a}$ does not intersect the plane $2x - 3y + z = 0$ then a must equal

A. 2 **B**. 3 **C**. 4 **D**. 5 **E**. None of these

ANS: The line must be orthogonal to the normal vector of the plane. Thus, we require that $\langle 2, -3, 1\rangle \bullet \langle 2, 3, a\rangle = 0$. Thus we want $a = 5$ and the correct answer is $\boxed{\mathbf{D}}$.

(c) Suppose that $f(x, y) = 5x^2 + 5xy^2$. Then the directional derivative of f at the point $(1, -1)$ in the direction of the vector $\mathbf{v} = \langle 3, 4\rangle$ is ...

A. -1 **B**. 0 **C**. 1 **D**. 2 **E**. None of these

ANS: Here $\mathbf{u} = \frac{1}{5}\langle 3,4\rangle$ and $D_{\mathbf{u}}f(1,-1) = \nabla f(1,-1) \bullet \mathbf{u} = \frac{1}{5}\langle 15,-10\rangle \bullet \langle 3,4\rangle = 1$. Therefore the correct answer is $\boxed{\mathbf{C}}$.

(d) If $f(x,y) = \ln(xy)$, then the linearization of f near $(1,1)$ is given by $L(x,y) = \dots$. (Recall that the linearization is the function whose graph is the tangent plane at $(1,1,0)$.)

A. $1+x+y$ **B**. $x+y$ **C**. $2+x+3y$ **D**. $\dfrac{1}{x}+\dfrac{1}{y}$

E. None of These

ANS: We have $f_x(x,y) = \frac{1}{x}$ and $f_y(x,y) = \frac{1}{y}$. Thus $L(x,y) = f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1) = x+y-2$. The correct answer is $\boxed{\mathbf{E}}$.

(e) The area of the parallelogram spanned by the vectors $\mathbf{v} = \langle 1,-1,2\rangle$ and $\mathbf{w} = \langle 2,0,1\rangle$ is ...

A. $\sqrt{11}$ **B**. $\sqrt{12}$ **C**. $\sqrt{13}$ **D**. $\sqrt{14}$ **E**. None of These

ANS: This is just $|\mathbf{v}\times\mathbf{w}| = |\langle -1,3,2\rangle| = \sqrt{14}$. Thus the answer is $\boxed{\mathbf{D}}$.

(f) Suppose that $\mathbf{r}$ is a vector-valued function of t. Then $\dfrac{d}{dt}(\mathbf{r}\times\mathbf{r})$ is

A. A unit vector normal to $\mathbf{r}$.

B. Positive when $|\mathbf{r}|$ is increasing and negative when $|\mathbf{r}|$ is decreasing.

C. The zero vector. **D**. The number zero. **E**. None of these.

ANS: Since $\mathbf{r}\times\mathbf{r}$ is constantly equal to the zero vector, the derivative is also constantly equal to the zero *vector*. The correct answer is $\boxed{\mathbf{C}}$.

NAME : ____________________

SECTION : (circle one) Groszek Mileti Williams

Math 11

20 October 2008

Exam I

Instructions: This is a closed book, closed notes exam. You are not to provide or receive help from any outside source during the exam except that you may ask any of the instructors for clarification. You have two hours and you should attempt all 7 problems.

- Wait for signal to begin.
- ***Print*** your name in the space provided and circle your instructor's name.
- Sign the FERPA release below *only if* you wish your exam returned via the homework boxes.
- Calculators or other computing devices are not allowed.
- Except for problems #6 and #7, you must show your work and justify your assertions to receive full credit.
- *Place your final answer in the space provided*!

FERPA RELEASE: Because of privacy concerns, we are not allowed to return your graded exams to the homework boxes without your permission. If you wish us to return your exam to your homework box, please sign on the line indicated below. Otherwise, you will have to pick your exam up in your instructor's office after lecture in which the exams are returned.

SIGN HERE: ____.

Problem	Points	Score
1	15	
2	10	
3	10	
4	15	
5	10	
6	16	
7	24	
Total	100	