11/18/10
Barnett

MATH 11 WORKSHEET : Curl & Div

Consider $\vec{F}(x,y,z) = (x^2y,\ 1,\ xyz^2)$ $\boxed{\text{vector field in } \mathbb{R}^3}$

A) Compute i) $\operatorname{div} \vec{F} =$

ii) $\operatorname{curl} \vec{F} =$

iii) Is $\vec{F}$ conservative?

B) Now compute div of curl $\vec{F}$ from your answer ii) above:

$\operatorname{div} \operatorname{curl} \vec{F} =$

Is this coincidence? Let's see by using general $\vec{F} = (P, Q, R)$.

Write $\operatorname{curl} \vec{F} = (R_y - Q_z, \ \ldots \qquad)$

Then take $\operatorname{div} \operatorname{curl} \vec{F} = \ldots$

(look for cancellation!)

If time... C) Compute $\operatorname{div} \operatorname{grad} f = \vec{\nabla} \cdot \vec{\nabla} f = \nabla^2 f$ for $f(x,y,z) = \dfrac{1}{\sqrt{x^2+y^2+z^2}}$

ie, $\frac{1}{r}$

—— SOLUTIONS ——

11/15/10
Barnett

MATH 11 WORKSHEET : Curl & Div

Consider $\vec{F}(x,y,z) = \left(x^2y,\ 1,\ xyz^2\right)$ vector field in $\mathbb{R}^3$

A) Compute i) $\operatorname{div}\vec{F} = P_x + Q_y + R_z = 2xy + 0 + 2xyz$

ii) $\operatorname{curl}\vec{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\times(P,Q,R)$

$= \left(xz^2,\ -yz^2,\ -x^2\right)$

iii) Is $\vec{F}$ conservative? no since conservative $\Rightarrow$ $\operatorname{curl}\vec{F} \equiv \vec{0}$

B) Now compute div of curl $\vec{F}$ from your answer ii) above:

$\operatorname{div}\operatorname{curl}\vec{F} = \frac{\partial}{\partial x}(xz^2) + \frac{\partial}{\partial y}(-yz^2) + \frac{\partial}{\partial z}(-x^2) = z^2 - z^2 \equiv 0$

Is this coincidence? no! Let's see by using general $\vec{F} = (P, Q, R)$.

Write $\operatorname{curl}\vec{F} = (R_y - Q_z,\ P_z - R_x,\ Q_x - P_y)$

Then take $\operatorname{div}\operatorname{curl}\vec{F} = \ldots (R_y - Q_z)_x + (P_z - R_x)_y + (Q_x - P_y)_z$

$= \cancel{R_{yx}} - \cancel{Q_{zx}} + \cancel{P_{zy}} - \cancel{R_{xy}} + \cancel{Q_{xz}} - \cancel{P_{yz}} \equiv 0$ [look for cancellation!]

If time... C) Compute $\operatorname{div}\operatorname{grad} f = \vec{\nabla}\cdot\vec{\nabla}f = \nabla^2 f = f_{xx} + f_{yy} + f_{zz}$ for $f(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$ ie, $\frac{1}{r}$

$f_x = 2x\left(-\frac{1}{2}(x^2+y^2+z^2)^{-3/2}\right) = \frac{-x}{r^3}$ so $\vec{\nabla}f = -\frac{\vec{r}}{r^3}$

prod. rule

$f_{xx} = -\frac{1}{r^3} - x\left(-\frac{3}{2}r^{-5}\cdot 2x\right) = \frac{3x^2 - r^2}{r^5}$ (similar for y, z).

$f_{xx} + f_{yy} + f_{zz} = \frac{1}{r^5}\left(3x^2 - r^2 + 3y^2 - r^2 + 3z^2 - r^2\right) = \frac{3r^2 - 3r^2}{r^5} \equiv 0$ (everywhere except $\vec{r} = \vec{0}$)

So $f(\vec{r}) = \frac{1}{r}$ is a solution to Laplace's eqn. $\nabla^2 f = 0$ in $\mathbb{R}^3 \setminus \{\vec{0}\}$ "fundamental soln."