Math 11
Fall 2007
Practice Problems

Here are some problems on the material we covered since the second midterm. This collection of problems is not intended to mimic the final in length, content, or difficulty.

The final exam will concentrate on material covered since the second midterm, but there will also be problems on earlier material.

1. True or False:

   (a) The function

   $$\vec{r}(t) = \vec{a} + t(\vec{b} - \vec{a}) \qquad 0 \le t \le 1$$

   parametrizes the straight line segment from $\vec{a}$ to $\vec{b}$.

   (b) If the coordinate functions of $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$ have continuous second partial derivatives, then $curl(div(\vec{F}))$ equals zero.

   (c) Putting together the two different vector forms of Green's Theorem, we can see that if D is a region satisfying the hypotheses of the theorem, and P and Q are functions satisfying the hypotheses of the theorem, we must have

   $$\int_{\partial D} (P, Q) \cdot \hat{T}\, ds = \int_{\partial D} (P, Q) \cdot \hat{N}\, ds$$

   Here $\hat{T}$ denotes the unit tangent vector to a curve, and $\hat{N}$ the unit normal vector, so the integral on the left is the usual line integral of $\vec{F} = (P, Q)$ along ∂D, and the integral on the right is the integral representing the flux of $\vec{F} = (P, Q)$ across ∂D.

   (d) For any vector field $\vec{F}$ in $\mathbb{R}^3$ all of whose coordinate functions have continuous first and second partial derivatives, we have that $div(curl(\vec{F})) = 0$.

   (e) If the vector field $\vec{F}$ is conservative on the open region D then line integrals of $\vec{F}$ are path-independent on D, regardless of the shape of D.

(f) If $\vec{F}$ is any vector field, then $curl(\vec{F})$ is a conservative vector field.

2. (a) Find a potential function f for the vector field

$$\vec{F}(x, y) = (2x + 2y,\ 2x + 2y).$$

A potential function is just a function f such that $\vec{F} = \nabla f$.

(b) Verify the Fundamental Theorem of Line Integrals for $\displaystyle\int_C \vec{F} \cdot d\vec{r}$ in the case

$$\vec{F}(x, y) = (2x + 2y,\ 2x + 2y)$$

and C is the portion of the positively oriented circle $x^2 + y^2 = 25$ from $(5, 0)$ to $(3, 4)$.

3. Find $\int_C \vec{F}(x, y)\, d\vec{r}$ where

$$\vec{F}(x, y) = \left(ye^{xy} + \cos x,\ xe^{xy} + \frac{1}{y^2 + 1} \right)$$

and C is the portion of the curve $y = \sin x$ from $x = 0$ to $x = \dfrac{\pi}{2}$.

4. The temperature at a point in space is given by the function

$$T(x, y, z) = z^2 - xy.$$

Heat flows from regions of high temperature to regions of low temperature, and the rate of heat flow is proportional to the rate at which temperature changes. That is, heat flow (in appropriate units) is given by

$$\vec{F}(x, y, z) = -\nabla T(x, y, z).$$

The rate at which heat flows across a surface S is given by the flux of the heat flow $\vec{F}$ across S,

$$\iint_S \vec{F} \cdot d\vec{S}.$$

If S is the surface given in cylindrical coordinates by

$$z = \theta \qquad r \leq 1 \qquad 0 \leq \theta \leq 2\pi$$

oriented so the unit normal vector slants upwards, find the rate at which heat flows across S.

Don't try anything fancy here. Just parametrize the surface and compute the flux.

5. Find
$$\iint_S \vec{F} \cdot d\vec{S}$$
where S is the conical surface
$$z^2 = x^2 + y^2 \qquad 0 \le z \le 1$$
oriented so the unit normal vector slants downwards, and $\vec{F}$ is the function
$$\vec{F}(x, y, z) = (x + \tan^{-1}(y^2),\ -y + \sec(x + z),\ z^2).$$
Note that S is not a closed surface. Nevertheless, there is a better way to do the problem than brute force.

6. Let C be the curve consisting of the line segments from $(0, 0)$ to $(1, 1)$ to $(0, 1)$ and back to $(0, 0)$. Find the value of
$$\int_C xy\, dx + \sqrt{y^2 + 1}\, dy.$$

7. Let $\vec{F}(x, y) = (e^x \sin y + 3y, e^x \cos y + 2x - 2y)$ and $\phi(x, y) = e^x \sin y + 2xy - y^2$.
   (a) Find $\nabla\phi(x, y)$.
   (b) Compute
   $$\int_C \vec{F} \cdot d\vec{r},$$
   where C is the positively oriented ellipse $4x^2 + y^2 = 4$. (Hint: make use of part (a) by comparing $\vec{F}$ and $\nabla\phi$.)

8. Evaluate the line integral of the function
$$F(x, y, z) = \left\langle x^2y^3,\ e^{xy+z},\ x + z^2 \right\rangle$$
around the circle $x^2 + z^2 = 1$ in the plane $y = 0$, oriented counterclockwise as viewed from the positive y-direction.

9. Compute the flux of the vector field

$$\vec{F}(x, y, z) = \langle 2x, y, 3z \rangle$$

outward through the sphere of radius 36 centered at the point $(1, 2, -1)$.

10. Let R be the region in the xy-plane above the x-axis and below the curve C parametrized by $\vec{r}(t) = \langle 1 + t^3, t - t^2 \rangle$ for $t \in [0, 1]$.

    (a) Sketch the region R. (Just do the best you can.)

    (b) Use Green's Theorem to express the area of R as a line integral.

    (c) Compute the area of R by evaluating your line integral from part (b).

11. Consider the vector field $\vec{F}(x, y, z) = \langle y + z, x - z, zy \rangle$.

    (a) Is $\vec{F}$ conservative? Why not?

    (b) Let C be any positively oriented simple closed curve in the xy-plane. Show that $\int_C \vec{F} \cdot d\vec{r} = 0$. (Hint: treat the region D in the xy-plane bounded by C as a surface and apply Stokes's Theorem.)

12. Show that if $\vec{F} = (F_1,\ F_2)$ is a vector field on $\mathbb{R}^2$ such that, on all of $\mathbb{R}^2$, the component functions F_1 and F_2 have continuous partial derivatives and

$$\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} = 0,$$

then the flux integral

$$\int_C \vec{F} \cdot \hat{N}\, ds$$

is path-independent. That is, if C_1 and C_2 are two piecewise smooth curves with the same endpoints, then

$$\int_{C_1} \vec{F} \cdot \hat{N}\, ds = \int_{C_2} \vec{F} \cdot \hat{N}\, ds.$$

If it helps, you may assume C_1 and C_2 do not cross, or have any points in common except their endpoints.

In this problem $\hat{N}$ denotes the unit normal vector to the curve. Hint: Use the second version of Green's Theorem; see page 1103 of the textbook.