Math 116. Lec 11. Th 10/30. (1)

Plan:
defn. S, D.
change of var.
jump relations.
DLP formulation
D cont kernel and ...

- Change of var: $g: \partial\Omega \to \mathbb{R}$ ie func on bdry.

$$\int_{\partial\Omega} g(y)\, ds_y = \int_0^{2\pi} g(y(t)) \underbrace{|y'(t)|}_{\text{speed func.}}\, dt$$

per. trap. quad.

$$\approx \frac{2\pi}{n} \sum_{j=1}^{n} g(y(t_j))\, |y'(t_j)|$$

- Single-layer operator: given bdry func σ, $(S\sigma)(x) := \int_{\partial\Omega} \Phi(x,y)\, \sigma(y)\, ds_y$ $\quad x \in \mathbb{R}^2$

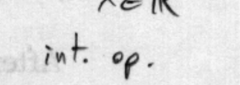
int. op.

interpret: σ = charge density
$(S\sigma)(x)$ = potential due to this charge (SLP)

(recall charge density on $[-1,1]$, Cheby).

- Double-layer op: given τ (dipole density), $(D\tau)(x) := \int_{\partial\Omega} \frac{\partial\Phi(x,y)}{\partial n_y}\, \tau(y)\, ds_y$ $\quad x \in \mathbb{R}^2$

potential due to dipoles. (DLP) : interp:



GRF states, $x \in \Omega$, $u(x) = (S\sigma)(x) + (D\tau)(x)$ if choose $\sigma = u_n|_{\partial\Omega}$, $\tau = -u|_{\partial\Omega}$

interior from bdry vals.

Cor 1). Choose $\Omega = B(x;r)$, then $u(x) = \ln r \underbrace{\int u_n(y)\, ds_y}_{\text{zero flux}} - \left(\frac{-1}{2\pi r}\right) \int u(y)\, ds_y$ = mean value on ∂B (MVT).

2) Max & min of harm. func. must occur on $\partial\Omega$ (unless its u = const)

pf: suppose max at $x \in \Omega$, $\exists B(x;r)$ with $r > 0$, but $u(x)$ = mean over ∂B, contradiction unless u = const.

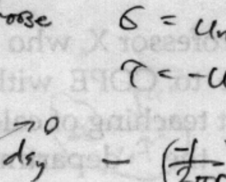
For min use $-u$.

3) Dirichlet BVP $\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega \end{cases}$ has at most 1 soln.

pf: suppose u, v solns, then $u - v = 0$ on $\partial\Omega$, by max principle $u - v \equiv 0$ in Ω

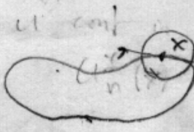
Peculiarities of LP's as approach $\partial\Omega$: Matlab show $u = S\sigma$ for $\sigma \equiv 1$, has const-sized kink at $\partial\Omega$: u cont. ∇u discont, jump due to surface charge

skip $\left[\begin{array}{l} \text{for } x \in \partial\Omega \text{ define } u^{\pm}(x) = \lim_{h \to 0^+} u(x \pm h\hat{n}_x) \\ u_n^{\pm}(x) = \lim_{h \to 0^+} \hat{n}_x \cdot \nabla u(x \pm h\hat{n}_x) \end{array}\right]$



Thm: let $\partial\Omega$ be C^1 cont. (ie $y(t) \in C^1$), $\sigma \in C(\partial\Omega)$, $u = S\sigma$

Then i) u cont in $\mathbb{R}^2$; for $x \in \partial\Omega$ $u(x) = \int_{\partial\Omega} \Phi(x,y)\, \sigma(y)\, ds_y$ exists as improper int.

Pf: (sketch)  choose ball $r>0$ s.t. 1 single connected piece of $\partial\Omega$ falls inside any $B(x;r)$ $\leq \delta r$.

and can be projected onto tangent plane with Jacobean at most 2. (can always be done)

Then $\forall x \in \mathbb{R}^2$, $\quad (S\sigma)(x) = \underbrace{\int_{\partial\Omega \cap \mathbb{R}^2\setminus B(x;r)} \Phi(x,y)\,\sigma(y)\,ds_y}_{\text{cont. wrt } x} + \int_{\partial\Omega \cap B(x;r)} \Phi(x,y)\,\sigma(y)\,ds_y$

$\leq 2 \underbrace{\int_{-r}^{r} \frac{1}{2\pi}|\ln|s||\,ds}_{\lim_{r\to 0} \text{ is } 0} \cdot \sup_{x\in\partial\Omega}|\sigma|$ use: $|y(s)-y(0)| \geq |s|$

so $S\sigma$ is unif limit of seq. of continuous funcs. in $\mathbb{R}^2$.

• SLP cont. since $\ln|r|$ integrable on a line (weakly singular).

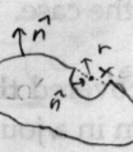
Thm (unit DLP): Let Ω be C^1 cont.

$$\int_{\partial\Omega} \frac{\partial\Phi(x,y)}{\partial n_y}\,ds_y = \begin{cases} -1 & x\in\Omega \\ -1/2 & x\in\partial\Omega \\ 0 & x\in\mathbb{R}^2\setminus\overline{\Omega} \end{cases}$$

Test in HW5.

Pf. $x\in\Omega$ GRF $u=-1$

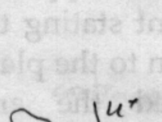
$x\in\mathbb{R}^2\setminus\overline{\Omega}$ $\Phi(x,\cdot)$ harm in Ω $\Rightarrow$ is just zero-flux statement.

$x\in\partial\Omega$:  write zero-flux since $\Phi(x,\cdot)$ harm. in $\Omega\setminus\overline{B(x;r)}$ $\quad 0 = \int_{\partial\Omega\cap\mathbb{R}^2\setminus B(x;r)} \frac{\partial\Phi(x,y)}{\partial n_y}\,ds_y + \int_{\partial B(x;r)\cap\Omega} \frac{\partial\Phi}{\partial n_y}(x,y)\,ds_y$

as $r\to 0$ this tends to $\frac{1}{2}\int_{\partial B(x,r)} \frac{\partial\Phi}{\partial n_y}(x,y)\,ds = \frac{1}{2}\cdot 2\pi r \cdot \frac{1}{2\pi r} = \ldots$ since $\hat n$ towards x

• note: says DLP has jump in value of 1 if density $\tau = 1$.

Define: $u^{\pm}(x) := \lim_{h\to 0^+} u(x \pm h\hat n)$

$u_n^{\pm}(x) := \lim_{h\to 0^+} \hat n_x \cdot \nabla u(x \pm h\hat n_x)$ 

then expect DLP has $u^+(x) - u^-(x) = \tau(x)$ .... true:

Thm (jump relations): Let $\partial\Omega$ be C^1 cont, $\sigma,\tau \in C(\partial\Omega)$ and $u = S\sigma$, $v = D\tau$, then for $x\in\partial\Omega$:

i) $u^{\pm}(x) = \int_{\partial\Omega} \Phi(x,y)\,\sigma(y)\,ds_y$ (no jump)

ii) $u_n^{\pm}(x) = \int \frac{\partial\Phi(x,y)}{\partial n_x}\,\sigma(y)\,ds_y \mp \frac{1}{2}\sigma(x)$ (jump is $-\sigma$) note!

iii) $v^{\pm}(x) = \int \frac{\partial\Phi(x,y)}{\partial n_y}\,\tau(y)\,ds_y \pm \frac{1}{2}\tau(x)$ (jump is τ) $\to$ dipole density causes potential jump

iv) $v_n^{\pm}(x) = \int \frac{\partial^2\Phi(x,y)}{\partial n_x \partial n_y}\,\tau(y)\,ds_y$ (no jump)

integrals are improper (since not defined $y=x$), but singularities integrable.

proofs: Colton–Kress, a bit technical.

So if think of S, D as integral ops on $C(\partial\Omega)\to C(\partial\Omega)$, and def. T w/ kernel $\frac{\partial^2\Phi(x,y)}{\partial n_x \partial n_y}$,

on $\partial\Omega$ have, $\quad u = S\sigma$

$u_n^{\pm} = (D^T \mp \frac{1}{2})\sigma$

$v^{\pm} = (D \pm \frac{1}{2})\tau$

$v_n^{\pm} = T\tau$

note T has strongly-singular kernel $\sim \frac{1}{x-y}$

JR(iii) is integral eqn in $C(\partial\Omega)$ for τ given v^- (bdry values approaching from inside):

$$(\mathcal{D} - \tfrac{I}{2})\tau = v^- \quad \text{or} \quad (I - 2\mathcal{D})\tau = -2v^- \quad 2^{nd}\text{-kind IE}$$

Thm: if $\tau \in C(\partial\Omega)$ a soln to $(I - 2\mathcal{D})\tau = -2f$ $(*)$, then $v = \mathcal{D}\tau$ solves $\Delta v = 0$ in Ω, $v = f$ on $\partial\Omega$ (interior Dirichlet BVP)

follows from JR(iii)

$\boxed{\text{got here}}$

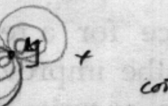
Numerical method: Nyström on $(*)$ to get $\tau(t_j)$ at nodes, then use these nodes to approx. $v(x)$ for all $x \in \Omega$ needed

Adv: reduced 2d to 1d problem! V. small lin. system. Don't need to compute all $x \in \Omega$ unless want.

Disadv: [illegible]

We can say more: claim kernel of $\mathcal{D}$ is <u>continuous</u> for C^2 domains (ie no corners) $\Rightarrow$ $\mathcal{D}$ compact op. $\Rightarrow$ $(*)$ unique soln. if 1 not an eigval of $2\mathcal{D}$.

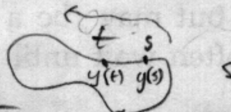
intuitively:



contours of $\frac{\partial \Phi(x,y)}{\partial n_y}$

If $\partial\Omega$ has well-defined curvature, $x \in \partial\Omega$ approaches on one of these circles.

proof:



$s, t \in [0, 2\pi)$

parametrize $\partial\Omega$ $y(t) \in \mathbb{R}^2$ $\quad$ $\Omega \subset C^2$ means $\left.\begin{array}{l}\dot{y}(t) = \frac{d}{dt} y \\ \ddot{y}\end{array}\right\}$ continuous (bnded) vector funcs.

Also demand $|\dot{y}(t)| > 0 \ \forall t$ - speed nonvanishing.

$\mathcal{D}$'s kernel $\displaystyle k(s,t) = \frac{1}{2\pi} \frac{\hat{n}(t) \cdot (y(s) - y(t))}{|y(s) - y(t)|^2}$ (last lec).

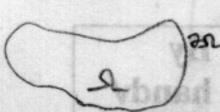
cont. for $s \neq t$ since top & bottom are.

$\lim_{s \to t} k(s,t)$ need l'Hôpital's rule twice:

$\frac{d}{ds}\,\text{top} = \hat{n}(t) \cdot \dot{y}(s)$, $\quad \frac{d^2}{ds^2}\,\text{top} = \hat{n}(t) \cdot \ddot{y}(s) \xrightarrow{\lim} \hat{n}(t) \cdot \ddot{y}(t)$

[finish & debug signs from 2006 notes!]

~~Lec.~~ 12 & 13. questionnaire. ① 11/7/08.



BVP $\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega \end{cases}$ ← Dirichlet 'data'

Double-layer operator: given funct on $\partial\Omega$, for $x \in \partial\Omega$, $(\mathcal{D}\tau)(x) := \int_{\partial\Omega} \frac{\partial \Phi}{\partial n_y}(x,y)\,\tau(y)\,dy$. If τ a soln to $(I - 2\mathcal{D})\tau = -2f$ (BIE)

then $u = \mathcal{D}\tau$ solves BVP, so is the unique soln.
check: • JR iii this DLP has correct bdry vals, approaching from inside.
• it is harmonic (can move Δ inside integral)

Num. meth: Nyström on (BIE) to get $\tau(t_j)$ at nodes, then use same nodes to approx $u(x)$ for all $x \in \Omega$

Adv: reduced 2d to 1d prob, much fewer degrees of freedom (size of linear system).
no geometric complexity (no meshing, etc).
highly accurate, spectral convergence (domains w/ analytic $\partial\Omega$, analytic data f)

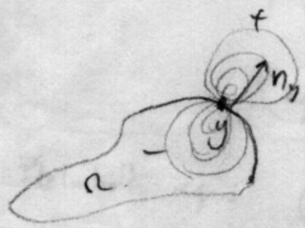
Disadv: linear sys is dense (direct discretization of BVP gives large sparse system)
evaluating soln. near bdry requires care

Can say more about $\mathcal{D} : C(\partial\Omega) \to C(\partial\Omega)$: Thm: kernel continuous for C^2 domains.

Cor: $\Rightarrow \mathcal{D}$ compact op.

~~⇒ (BIE) has unique soln. if 1 not eigval of $2\mathcal{D}$.~~ $\in \mathbb{R}^2$
~~⇒ existence proof for BVP (historical; there are others)~~

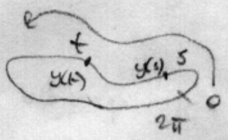
Why cont?



← contours of $\frac{\partial \Phi(\cdot,y)}{\partial n_y}$ are circles passing through y, tangent to $\partial\Omega$

If $\partial\Omega$ has cont. curvature, $x \in \partial\Omega$ approaches y on one of these.

pf:



parametrize $\partial\Omega$ by $y : [0, 2\pi) \to \mathbb{R}^2$ vector func. $\Omega \in C^2$ means $\left.\begin{array}{l} \dot{y}(t) = \frac{d}{dt} y \\ \ddot{y} \end{array}\right\}$ cont ($\Rightarrow$ bnded) vec func

Demand $|\dot{y}(t)| > 0 \ \forall t$: nonvanishing speed.

kernel of $\mathcal{D}$ is $k(s,t) = \frac{1}{2\pi} \frac{\hat{n}(t) \cdot (y(s) - y(t))}{|y(s) - y(t)|^2}$

$k(s,t) = \frac{1}{2\pi} \frac{\cos\theta}{r}$

top & bottom cont. $\Rightarrow$ cont $\forall s \neq t$.

$\lim_{s \to t} k(s,t)$ top & bottom vanish $\Rightarrow$ l'Hôpital: $\frac{d}{ds}$ top $= \hat{n}(t) \cdot \dot{y}(s) \to 0$ also!

$\frac{d^2}{ds^2}$ top $= \hat{n}(t) \cdot \ddot{y}(s) \xrightarrow{\lim} \hat{n}(t) \cdot \ddot{y}(t)$

$\frac{d}{ds}$ bottom $= 2\dot{y}(s) \cdot [y(s) - y(t)]$, $\frac{d^2}{ds^2}$ bottom $= 2|\dot{y}(s)|^2 \xrightarrow{\lim} 2|\dot{y}(t)|^2$

So $\lim_{s \to t} k(s,t) = \frac{1}{4\pi} \frac{\hat{n}(t) \cdot \ddot{y}(t)}{|\dot{y}(t)|^2} = -\frac{\kappa(t)}{4\pi}$ κ = curvature $\left(\begin{array}{ll} >0 & \text{convex} \\ <0 & \text{concave} \end{array}\right) = \frac{1}{\text{rad. of curvature}}$

Need for $k(t_j, t_j)$ in Nyström.

Thm: (LIE 6.20) $I - 2D$ is injective, ie trivial nullspace, ie 1 not eigval of $2D$. (proof many steps)

Cor: by Fredholm alternative in $C(\partial\Omega)$, $(I-2D)\tau = -2f$ has unique soln. τ
$\Rightarrow$ soln. to BVP, $u = D\tau$, exists $\forall f \in C(\partial\Omega)$.

Historically such BIEs, Fredholm alt., were first such proof ... existence.

Other BVPs for Laplace eqn: i) int. Neumann
$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u_n = g & \text{on } \partial\Omega \end{cases}$$

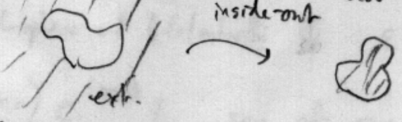
zero-flux $\underbrace{\int_{\partial\Omega} u_n \, ds = \int_{\partial\Omega} g \, ds = 0}$ necessary — turns out sufficient condition for soln to exist

non-unique: if u soln, so is $u + c$.

ii) exterior Dirichlet
$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^2 \setminus \overline{\Omega} \\ u = f & \text{on } \partial\Omega \\ u(x) = O(1) & \text{as } |x| \to \infty, \text{ uniformly in angle. 'harmonic at } \infty\text{'.} \end{cases}$$

$u = D\tau$ harm. in $\mathbb{R}^2 \setminus \overline{\Omega}$, satisfies 'harm. at ∞' cond.
$\Rightarrow$ use JR3: $u^+ = (D + \tfrac{1}{2})\tau = f$
ie $(I + 2D)\tau = 2f$
if τ solves this BIE then $u = D\tau$ solves ext. BVP.

has unique soln. $\forall f$.
Uniqueness Follows from $\tilde{u}(x) := u\left(\frac{x}{|x|^2}\right)$ "Kelvin xform of u"
also being harmonic in $\tilde{\Omega} = \{x : \frac{x}{|x|^2} \in \mathbb{R}^2 \setminus \overline{\Omega}\}$



ext. $\xrightarrow{\text{inside-out}}$ interior problem, unique, exists.

eg. Folland PDE book.

Timeharmonic Waves. (11/6/08)

obey; Helmholtz eqn. $(\Delta + k^2) u = 0$

k = wavenumber $= \frac{2\pi}{\text{wavelength}}$

another example of elliptic 2nd-order homog. PDE
$$\sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{j=1}^{n} b_j \frac{\partial u}{\partial x_j} + cu = 0$$
where a_{ij}, b_j, c can depend on x.
matrix $(a_{ij}(x))_{i,j=1,\dots,n}$ is positive-definite $\forall x$.

Wave eqn. $k = \frac{\omega}{c}$ ← freq, ← speed. eg u = acoustic pressure w/ $e^{i\omega t}$ time-dep.
Schrodinger eqn $k^2 = \frac{2mE}{\hbar^2}$, u = wavefunc.

Interior BVP
$$\left.\begin{aligned} (\Delta + k^2) u &= 0 && \text{in } \Omega \\ u &= f && \text{in } \partial\Omega \end{aligned}\right\}$$ driving a cavity on its bdry.

unique solution unless $\left\{\begin{aligned} -\Delta u &= k^2 u && \text{in } \Omega \\ u &= 0 && \text{in } \partial\Omega \end{aligned}\right\}$ has nontrivial soln, ie k^2 is eigval E_j of $-\Delta$ w/ Dirichlet BCs.

Can show Δ^{-1} compact $\Rightarrow E_j$ discrete & limit pt. is ∞

we'll return to these 'Dirichlet eigenvalues' later

where ... $\begin{pmatrix} \geq 0 & \text{convex} \\ \leq 0 & \text{concave} \end{pmatrix}$. $= \frac{1}{\ }$

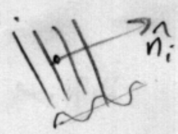
Exterior BVP.

$$\begin{cases} (\Delta + k^2) u^s = 0 & \text{in } \mathbb{R}^d \setminus \overline{\Omega} \qquad d = 2, 3, \cdots \\ u^s = f & \text{on } \partial\Omega \\ \lim_{r\to\infty} \underbrace{r^{\frac{d-1}{2}}}_{\sqrt{r} \text{ for } d=2} \left(\frac{\partial u^s}{\partial r} - iku^s \right) = 0 \end{cases}$$

Sommerfeld radiation condition: outgoing waves only.

Will show has unique soln. $\forall f \in C(\partial\Omega)$. (proof CK Thm 3-7)

Scattering of waves: say incident wave $u^i : \mathbb{R}^2 \to \mathbb{C}$ satisfies $(\Delta + k^2) u^i = 0$ in $\mathbb{R}^2$. eg. $u^i(x) = e^{ik\hat{n}_i \cdot x}$ plane wave.



Then if $\underbrace{f = -u^i|_{\partial\Omega}}_{\text{cancels out incident}}$, $u \equiv u^i + u^s$ solves $(\Delta + k^2) u = 0$ in $\mathbb{R}^2 \setminus \overline{\Omega}$, $u = 0$ on $\partial\Omega$

with obstacle generating only outgoing waves.

Solving Helmholtz BVP:
Fundamental soln.
$$\Phi(x,y) = \frac{i}{4} H_0^{(1)}(k|x-y|) \qquad d = 2.$$
$H_0^{(1)}$: outgoing Hankel func, a special func. – properties in Abramowitz & Stegun, www. – Matlab/numpy can evaluate.

As $r \equiv |x-y| \to 0$, $\Phi(x,y) = -\frac{1}{2\pi} \ln|x-y| + O(1)$, ie same sing. as for Laplace' eqn.

$\Rightarrow$ All jump relations same as before,

take-home msg: can replace Laplace w/ Helmholtz Φ & BIEs same as before! (HW6).

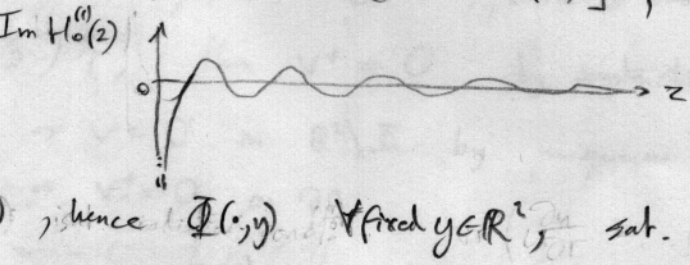
Where Hankel from? $u(\underbrace{r,\theta}_{x}) = f(kr) e^{im\theta}$ sep. of var.; fix $m \in \mathbb{Z}$ & find $f(z)$ sat. Helm Eqn.

$$(\Delta + k^2) u = \frac{1}{r} \partial_r (r \partial_r u) + \frac{1}{r^2} \partial_{\theta\theta} u - k^2 u = \left(k^2 f'' + \frac{kf'}{r} \right) e^{im\theta} - \frac{m^2 f}{r^2} e^{im\theta} - k^2 f e^{im\theta} = 0$$

$$\text{so } z^2 f'' + z f' + (z^2 - m^2) f = 0 \qquad \text{Bessel's eqn, } m^{\text{th}} \text{ order.}$$

$H_m^{(1)}(z)$ is a soln. w/ certain singularity at $z \to 0$ & outgoing.

$$\text{eg. } H_0^{(1)}(z) = \sqrt{\frac{2}{\pi z}} e^{i(z - \pi/4)} \left[1 + O\left(\frac{1}{z}\right) \right], \quad z \to \infty. \quad \text{asymptotic}$$



decaying complex exponential.

fact: $H_0^{(1)}(kr)$, hence $\Phi(\cdot, y)$ $\forall$ fixed $y \in \mathbb{R}^2$, sat. radiation condition