M116 | Lec. 4. 2^{nd} half. 10/6/08 (1)

Kress NA Ch. 8.

Interpolation: approximating a func f on $[a,b]$ by degree-n poly:

$$p(x) = \sum_{k=0}^{n} a_k x^k \quad \leftarrow \text{Lin. Indep. on } [a,b].$$

'Fit' the poly at $n+1$ distinct points (nodes): $x_j \in (a,b)$ $\quad p(x_j) = y_j \quad \leftarrow$ the y-values $f(x_j)$, $j = 0 \cdots n$.

$$\begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ \vdots & & \vdots \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{pmatrix} f(x_1) \\ \vdots \\ f(x_n) \end{pmatrix} \qquad M \in \mathbb{R}^{(n+1)\times(n+1)}$$

proved $\det M \neq 0$ in lec 1. $\Rightarrow$ soln. exists, unique.

Prop: $p = \sum_{k=0}^{n} y_k \ell_k$ where $\ell_k(x) = \prod_{j=0, j\neq k}^{n} \frac{x - x_j}{x_k - x_j}$, $k = 0 \cdots n$ are Lagrange basis (1794)

why?

pf: $\ell_k(x_i) = \begin{cases} \prod_{j \neq k} \frac{x_k - x_j}{x_k - x_j} = 1 & i = k \\ 0 & i \neq k \text{ since a factor of } (x_j - x_j) = 0 \end{cases} = \delta_{ki}$

so $p(x_i) = \sum y_k \delta_{ki} = y_i$ is a soln. ie Lagrange basis solves the problem.

- n large (>30) may cause stability probs. due to $\sup_{[a,b]} |\ell_k(x)|$ large.
- Newton 1676 realised a more practical method, 'divided diffs' $\leadsto$ we won't do.
- the map from func f to its unique poly approx p through $\{x_j\}$ is linear: $p = L_n f$

$L_n : C[a,b] \longrightarrow \mathbb{P}_n \quad \leftarrow$ space of degree-n polys.

If $p \in \mathbb{P}_n$ then $L_n p = p$ $\qquad$ so what kind of op. is L_n? projection. $L_n^2 = L_n$.

Error of interpolation $\quad L_n f - f \quad$ is a function.

Thm 8.10 Let $f \in C^{n+1}[a,b]$, then for each $x \in [a,b]$ there exists a ξ s.t.

$$f(x) - L_n f(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{j=0}^{n} (x - x_j)$$

$f \in C^k[a,b] \leftarrow$ means k-times continuously differentiable ie $f^{(k)} \in C$.

So, if you know $|f^{(n+1)}(x)| \leq C$ in $x \in [a,b]$, you can bound the error. 'error estimate' $\leftarrow$ means strict.

pf: trivial if $x = x_j$

Fix $x \neq x_j$, define $g(y) := f(y) - L_n f(y) - \prod_{j=0}^{n} (y - y_j) \frac{f(x) - L_n f(x)}{\prod_{j=0}^{n}(x - x_j)}$ $\quad y \in [a,b]$

Set $y = x_j$: $\quad 0 \quad 0 \quad \Rightarrow g(x_j) = 0$.

$y = x$: $\quad g(x) = 0 \quad$ so has $n+2$ zeros.

By Rolle's thm g' has $\geq n+1$ zeros.

etc: $g^{(n+1)}$ has ≥ 1 zero, call it ξ.

set $y = \xi$: $\quad 0 = f^{(n+1)}(\xi) - 0 \leftarrow$ since degree-n $\quad - (n+1)! \frac{f(x) - L_n f(x)}{\prod_{j=0}^{n}(x - x_j)}$ $\quad$ QED $\quad$ sneak

(WS) on Lagrange $\ell_j(x)$. MA6, Lec 5

Kress, NA. 10/8/① ②

Prove interp. error bnd thm 8.10. (state thm first).

· equi-spaced points are in general bad ; much better is to cluster pts. why?

→ show Mathematica nb applet : $f(x) = \frac{1}{1+25x^2}$ Runge. smooth, analytic

Note L^∞ bnd $\frac{\|f^{(n+1)}\|_\infty}{(n+1)!} h^{n+1}$ if nothing known beyond $x, x_0, \dots x_n \in [a,b]$

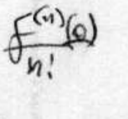
max. $n+1^{st}$ Taylor coeff. on $[a,b]$.

So, for what f expect problems?

Darboux : if f meromorphic, with simple poles, anal. in nei of O, Taylor series $f(z) = \sum_{n=0}^\infty a_n z^n$ $a_n = \frac{f^{(n)}(0)}{n!}$

(Henrici v.2 Thm 10-10a)

has $n\to\infty$ asymptotic $|a_n| \sim r\, d^{-n-1}$ where d is distance to nearest pole a, and r is residue of pole.



min. dist from interval to pole a.

So, $d := \min_{x\in[a,b]} |x-a| > h$ then L^∞ err $\sim r\left(\frac{h}{d}\right)^{n+1} \to 0$ as $n\to\infty$

pole far away. unif. exponential convergence

but if $d < h$, may have L^∞ err $\not\to 0$.

(pole nearby).   I will have it for you figure where poles of $\frac{1}{1+25x^2}$ are in $\mathbb{C}$.

Illustrates.. bad news : if construct seq of interp. operators L_n each with $\{x_j^{(n)}\}_{j=0}^n$ nodes,

Thm 8.17 (Faber) : for each seq. $\{x_j^{(n)}\}$ $\exists f \in C[a,b]$ st. $L_n f \not\to f$ unif. on $[a,b]$.

Good news : Thm 8.16 (Marcinkiewicz) for each $f \in C[a,b]$, $\exists$ seq. $\{x_j^{(n)}\}_{j=0}^n$ $n=0,1,\dots$ st. $L_n f \to f$ uniformly on $[a,b]$.

Why best to cluster points $\{x_j\}_{j=0}^n$ near ends of $[-1,1]$?   (Trefethen, Spec Meth Ch. 5)

$$\frac{1}{n+1}\ln\left|\prod_{j=0}^{n}(z-x_j)\right| = -\frac{1}{n+1}\sum_{j=0}^{n}\ln\frac{1}{|z-x_j|} = \text{electrostatic potential in } \mathbb{C} \text{ due to } n+1 \text{ points of re charges } -\tfrac{1}{n+1} \text{ at nodes.}$$

$(q_{n+1}(z))$ monomial in error bnd, we'd used $\le h^{n+1}$ bnd. $= \phi_{n+1}(z)$

so $|q_{n+1}| = e^{(n+1)\phi}$

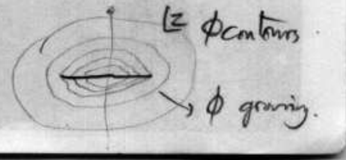
As $n\to\infty$ assume nodes tend to a density func. $\rho(x) > 0$ on $[-1,1]$, $\int_{-1}^1 \rho(x)\,dx = 1$. then $\phi_{n+1} \to \phi(z) = -\int_{-1}^1 \rho(x)\ln\frac{1}{|z-x|}\,dx$

Uniform nodes $\rho = 1/2$ so $\phi(z) = \frac{1}{2}\int_{-1}^1 \ln|z-x|\,dx = -\frac{1}{2}\mathrm{Re}\int_{z+1}^{z-1} \ln x\, dx$

$$= -\tfrac{1}{2}\mathrm{Re}\left[(z-1)\ln(z-1) - z+1 - (z+1)\ln(z+1) + z+1\right]$$

so $\phi(0) = -1$, $\phi(\pm 1) = -1 + \ln 2$ larger at ends

so $|q_{n+1}| \approx e^{(n+1)\ln 2}$ or 2^{n+1} times larger at ends.



Henrici book, Vol. 2 : Darboux Thm (Baby one).

illustrated by Example 1. Decomposing the generating function of the Fibonacci numbers into partial fractions, we obtain

$$\frac{1}{1-t-t^2}=\frac{1/\sqrt{5}}{(\sqrt{5}-1)/2-t}+\frac{1/\sqrt{5}}{(\sqrt{5}+1)/2+t}.$$

The first partial fraction has a pole at $t=t_1:=(\sqrt{5}-1)/2$, the second at $t_2=(-\sqrt{5}-1)/2$. We note that $|t_2|>|t_1|$.

Ignoring that the complete partial fraction decomposition is known, we write the foregoing in the form

$$\frac{1}{1-t-t^2}=\frac{1/\sqrt{5}}{(\sqrt{5}-1)/2-t}+g(t),$$

where about g we merely need to know that it is analytic for $|t|\leqslant\rho$ where $\rho>|t_1|$. Now the first term can immediately be expanded in a power series:

$$\frac{1/\sqrt{5}}{(\sqrt{5}-1)/2-t}=\frac{\sqrt{5}+1}{2\sqrt{5}}\sum_{n=0}^{\infty}\left(\frac{2t}{\sqrt{5}-1}\right)^n.$$

As to the power series of g, we know by the Cauchy estimate that its coefficients are bounded by $\mu\rho^{-n}$, where μ is a constant. It thus follows that

$$f_n=\frac{\sqrt{5}+1}{2\sqrt{5}}\left(\frac{2}{\sqrt{5}-1}\right)^n+O(\rho^{-n}),$$

or

$$f_n\sim\frac{1}{\sqrt{5}}\left(\frac{\sqrt{5}+1}{2}\right)^{n+1},\qquad n\to\infty.$$

We next consider a more general case:

simpler than Darboux Thm.

THEOREM 11.10a

Let the function p be meromorphic, with simple poles at the points t_m, $m=1,2,\ldots$, where $|t_{m+1}|>|t_m|$, $m=1,2,\ldots$, and let r_m be the residue at t_m. Then the coefficients p_n defined by

ie r_m = pole strength

$$p(t)=\sum_{n=0}^{\infty}p_n t^n$$

eg $p(z)=\sum_m r_m(t-t_m)^{-1}$

possess the following asymptotic expansion in terms of the asymptotic sequence $\{t_m^{-n}\}$:

$$p_n\approx-\sum_{m=1}^{\infty}\frac{r_m}{t_m^{n+1}},\qquad n\to\infty.\qquad(11.10\text{-}11)$$

ie if only one pole, t_1, then $p_n\sim-\frac{r_1}{t_1^{n+1}}$

Proof. For any positive integer m, the function

$$p(t)-\frac{r_1}{t-t_1}-\frac{r_2}{t-t_2}-\cdots-\frac{r_m}{t-t_m}$$

is analytic in $|t|<|t_{m+1}|$. Its nth Taylor coefficient,

$$p_n+\frac{r_1}{t_1^{n+1}}+\cdots+\frac{r_m}{t_m^{n+1}},$$

thus is $O(\rho^{-n})$ for $n\to\infty$, where ρ is any number $<|t_{m+1}|$. There follows

$$\lim_{n\to\infty} t_m^n\left[p_n+\frac{r_1}{t_1^{n+1}}+\cdots+\frac{r_m}{t_m^{n+1}}\right]=0$$

for $m=1,2,\ldots$. The formal series (11.10-11) thus satisfies property (B) of §11.9, which is equivalent to the statement of the theorem. ■

EXAMPLE 8

Let $\{p_n\}$ be the sequence of Taylor coefficients of $\Gamma(z)$ at $z=1$,

$$\Gamma(1+t)=\sum_{n=0}^{\infty} p_n t^n.$$

It is known that $p_0=1$, $p_1=-\gamma$ (the Euler constant); no simple formula for the general coefficient exists. However, an asymptotic expansion is easily found. The function $\Gamma(1+t)$ has simple poles at the points $t_m=-m$, $m=1,2,\ldots$, with residues $r_m=(-1)^{m-1}(m-1)!$; hence Theorem 11.10a yields

$$p_n\approx\sum_{k=1}^{\infty}(-1)^{n+k-1}\frac{(k-1)!}{k^{n+1}},\qquad n\to\infty.$$

It is easy to see by means of the ratio test that the above series diverges for every n. However, as an asymptotic series it has a definite meaning.

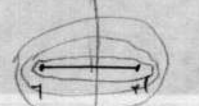
Theorem 11.10a can be extended to the case in which there are poles of order higher than 1, or where several poles have equal moduli. We leave these generalizations to the imagination of the reader and turn instead to the situation, also of frequent occurrence in practice, in which the generating function has singularities other than poles on the boundary of its disk of convergence. Because there is no partial fraction expansion in such cases, the simple device of subtracting singularities no longer works. However, asymptotic expansions can frequently be obtained by a method originally due to Darboux. It makes use of certain elementary properties of Fourier series.

Before stating Darboux' result in a simple special case, we recall from §11.9 that, for any complex number ν that is not an integer, the sequence of functions defined on the positive integers $n=1,2,\ldots$ by

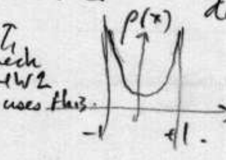
$$g_k(n):=\frac{(\nu-k)_n}{n!},\qquad k=0,1,2,\ldots \tag{11.10-12}$$

Is there a ρ that gives ϕ uniform in $(-1,1]$? 10/9. (2)

Solve electrostatics problem w/ $[-1,1]$ conducting (metal) !



analytic soln (complex analysis book): contours of ϕ are ellipses, $\rho(x) = \frac{1}{\pi\sqrt{1-x^2}}$ Chebychev density, cluster near ends.



$\phi(z) = -\ln 2$ const on $(-1,1]$.

so $|q_{n+1}| \approx \frac{1}{2^{n+1}}$ smallest can be uniformly.

$q_{n+1}(x)$:  roughly equi oscillating.

Can show that singularities of f can be arb. close to $[-1,1]$ & still get exponential conv. of L^∞ err. bnd for interp.
↳ a 'spectral method'?, ie error is K^{-n}, $K>1$. rather than n^p, $p>0$.

§9.1 quadrature.

want approx. $Q(f) := \int_a^b f(x)\,dx$

use $Q_n(f) := \sum_{k=0}^{n} w_k f(x_k)$ (weights; nodes in $[a,b]$)

Q, Q_n are linear functionals: $C(a,b) \to \mathbb{R}$.

Given nodes, what are good weights?

Choose 'interpolatory' weights, such that $Q_n(f) = \int_a^b (L_n f)(x)\,dx$ ie integrate the interpolation poly exactly.

use Lagrange basis $= \sum_{k=0}^{n} \underbrace{\int_a^b l_k(x)\,dx}\, f_k(x)$ fixes w_k for interpolatory quad.

Thm 9.2 given distinct nodes $\{x_j\}_{j=0}^n$ the above $\{w_k\}_{k=0}^n$ are the unique set which integrates all $p \in \mathbb{P}_n$ exactly

pf: $Q_n(p) = \int_a^b \underbrace{(L_n p)}_{p}(x)\,dx = \int_a^b p(x)\,dx$, exact. Unique since $\sum w_k f(x_k) = \sum w_k (L_n f)(x_k)$ (only vals at nodes) $= \int (L_n f)(x)\,dx$ if exact, ⇒ interpolatory, so ⟺ interp. exact

So, exact integration up to degree-n could be taken as defining feature.

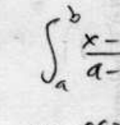
called 'Newton-Cotes' quad (sometimes assumes nodes equally spaced). Newton, 1600's.

Eg. $n=1$

$W_0 = \int_a^b l_0(x)\,dx = \int_a^b \frac{x-b}{a-b}\,dx = \frac{1}{2}(b-a) = \frac{h}{2}$

$W_1 =$ same.

so $Q_1(f) = h\,\frac{f(a)+f(b)}{2}$ trapezoid rule



Error anal. Thm 9.4 Let $f \in C^2(a,b]$ then $\underbrace{\int_a^b f(x)\,dx - Q_1(f)}_{E_1(f)} = -\frac{h^3}{12} f''(\xi)$ for some $\xi \in [a,b]$

pf. $E_1(f) = \int_a^b f(x) - L_1 f(x)\,dx = \int_a^b \underbrace{(x-a)(x-b)}_{\le 0}\, \underbrace{\frac{f(x) - L_1 f(x)}{(x-a)(x-b)}}\,dx$ continuous by l'Hôpital at endpts.

$= \underbrace{\frac{f(z) - L_1 f(z)}{(z-a)(z-b)}}_{= \frac{f''(\xi)}{2!} \text{ some } \xi} \underbrace{\int_a^b (x-a)(x-b)\,dx}_{\int_0^h x(x-h)\,dx = -\frac{1}{6}h^3}$ for some $z \in [a,b]$ by MVT for integrals.

$g \ge 0$, $f \in C$; $\int fg\,dx = g(z)\int f\,dx$. some z.

QED

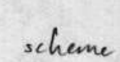
note

Quadrature: getting most accuracy w/ minimum # func. evals. (effort). [cont.]

We introduced Newton-Cotes scheme: given some nodes $\{x_j\}_{j=0}^n$ in $[a,b]$

there exists unique set of weights $\{w_j\}_{j=0}^n$ s.t. $Q_n(f) := \sum_{j=0}^n w_j f(x_j)$ exact for $f \in \mathbb{P}_n$

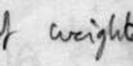
We tried it out for linear (n=1) eg. $[-1,1]$,



then split $[-1,1]$ into $\frac{2}{h}$ pieces of length h to get 'composite' rule with error $O(h^2)$ algebraic, convergence w/ effort

Rather than making composite,

To get higher order try eg. n=2



, get $Q_2(f) = \frac{1}{3}f(-1) + \frac{4}{3}f(0) + \frac{1}{3}f(1)$

Simpson's 1743 (Kepler 1612). $\{w_j\}$ can be solved by requiring exact integration for $1, x, x^2$.

- What if continue using n+1 equal-spaced nodes? n=3, 4, …

For $n>8$ get negative w_j's; higher $n \to$ exponentially large w_j of oscillating sign (since l_k basis were) = bad. (rounding errors become huge).

Convergence of quad schemes. (§9.2). consider seq $(Q_n)_{n=0}^{\infty}$ of schemes, $Q_n(f) := \sum_{j=0}^n w_j^{(n)} f(x_j^{(n)})$

Defn: (Q_n) conv. if $Q_n(f) \to Q(f) := \int_a^b f(x)\,dx$ as $n\to\infty$, $\forall f \in C[a,b]$. nice property

Thm (Szegö) Let (Q_n) be conv. for all polynomials p, and let $\sum_{j=0}^n |w_j^{(n)}| \le C \quad \forall n$. ← ie weights don't grow in size

Then (Q_n) convergent.

pf:

2 facts

i) $\mathbb{P}$ = poly's 'dense' in $C[a,b]$, meaning $\forall f \in C[a,b]$ & $\forall \varepsilon > 0$ ← no matter how small, $\exists p \in \mathbb{P}$ s.t. $\|f-p\|_\infty \le \varepsilon$ (Weierstrass)

ii) each Q_n is lin. op. $Q_n: C[a,b] \to \mathbb{R}$! w/ $|Q_n(f)| \le \|f\|_\infty \sum_{j=0}^n |w_j| \le C\|f\|_\infty$.

since C is largest blow-up factor in L^∞ (sup) norm, note equality poss. if f achieves max at each node. we say $\|Q_n\|_\infty = C$.

iii) We're done if can show: a seq. of bnded lin. ops which converges pointwise on dense subset ($\mathbb{P}$) converges pointwise everywhere. ($C[a,b]$)

ptwise conv? means for all funcs f in a set (either $C[a,b]$ or $\mathbb{P}$), $Q_n f \to Qf$ as $n\to\infty$ ← these are numbers.

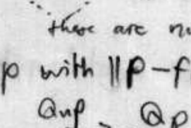
for any $\varepsilon>0$,

$$Q_nf - Qf = Q_nf \overbrace{- Q_np + Q_np}^{0!} \overbrace{- Qp + Qp}^{0!} - Qf$$

finding a p with $\|p-f\|_\infty \le \varepsilon$

Take abs val & tri ineq.

$$|Q_nf - Qf| \le \underbrace{|Q_nf - Q_np|}_{\le C\|f-p\|_\infty = C\varepsilon} + \underbrace{|Q_np - Qp|}_{\le \varepsilon \text{ for all } n>N, \text{ for some } N.} + \underbrace{|Qp - Qf|}_{\le (b-a)\varepsilon}$$



$$\le \underbrace{(C+1+b-a)}_{\text{fixed const.}}\varepsilon$$

So we can find an N s.t. $\forall n>N$, $|Q_nf - Qf|$ smaller than any positive #. QED

30 mins.

- This is "ε/3" argument from func. anal., converg.
- Szegö's Thm actually includes converse., which requires Principle of Uniform Boundedness (Banach–Steinhaus)

Why useful?

Corollary (9.11, Steklov): if (Q_n) conv. for all poly's, and $w_j^{(n)} \geq 0$; then (Q_n) convergent.

pf: $\|Q_n\|_\infty = \sum_{j=0}^{n} |w_j^{(n)}| = \sum_{j=0}^{n} w_j^{(n)} = Q_n(1) \xrightarrow{\text{conv. for poly's}} Q(1) = \int_a^b dx = b-a.$
(↑ nonneg.)

so there's a const s.t. $\|Q_n\|_\infty \leq C$, use thm.

Point: any family of quadr. scheme i) convergent for polys & ii) nonneg. weights is convergent ($\forall f \in C[a,b]$) & has no unnecessary roundoff error (cancellation)

⇒ composite trapezoid (last time) is conv. (n linear intervals)

but, Newton Cotes as $n \to \infty$ might not be.

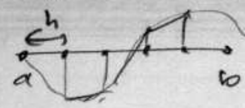
We now do better scheme, which will be convergent too!

5 mins.

→ Gaussian quadr. WS.

Cor : $E_1(f) \le \frac{h^3}{12} \|f''\|_\infty$

May split up longer interval $[a,b]$ into $[a, a+h]$ $[a+h, a+2h]$, ... $[b-h, b]$.



Composite trapezoid rule

$$\int_a^b f(x)dx \approx \frac{1}{h}\left[\frac{f(a)}{2} + f(a+h) + \cdots + \frac{f(b)}{2}\right]$$

Error $E(f) \le \frac{b-a}{h} \cdot E_1(f) = \underbrace{\frac{(b-a)}{12}\|f''\|_\infty}_{\text{prefactor}} h^2$ ← 2^{nd}-order convergence

is spectral? no! merely algebraic.

stopped

Lec. 6

To get exp. conv, fix $h = b-a$, one interval, increase order n.

[$n=2$] x_0 x_1 x_2 (a, b) 3 nodes. $Q_2(f) = h\left[\frac{h}{6}f(a) + \frac{2h}{3}f(\frac{a+b}{2}) + \frac{h}{6}f(b)\right]$ (w_0, w_1, w_2) Simpson's 1743 (Kepler 1612).

↓ as n incr, equal-spaced leads to large, oscillatory $(\pm)$ weights (since l_k's are too) = bad (rounding errors)

... ... Chebychev-spaced keeps weights $O(1)$ = good

9.3 Gaussian quad: do WS straight in.

Let's show this. ↳ got $n=2$ integrating degree-5 exactly, for general n, possible to get $2n+1$ exactly!

use orthog. $f \perp g$, means $(f,g) = \int_a^b f(x)g(x)\,dx = 0$.

Defn: $(n+1)$-node Gauss. quad. integrates $\mathbb{P}_{2n+1}$ exactly.

Lemma 9.13 Let $x_0 \cdots x_n$ be distinct nodes of a Gauss.-quad.

then $q_{n+1}(x) := \prod_{j=0}^{n}(x-x_j) \perp p, \quad \forall p \in \mathbb{P}_n$

pf: $q_{n+1}p \in \mathbb{P}_{2n+1}$ so $\int_a^b q_{n+1}(x)p(x)\,dx \overset{\text{Gauss.}}{=} \sum_{k=0}^{n} w_k q_{n+1}(x_k) p(x_k) = 0$. ($q_{n+1}(x_k)$ = 0 nodes.)

Lemma 9.14 : Converse of this holds. : if $\{x_j\}$ nodes sat. $q_{n+1} \perp \mathbb{P}_n$, it's Gauss. quad.

pf: interpolatory quad has $\sum w_k f(x_k) = \int_a^b (L_n f)(x)\,dx \quad \forall f \in C[a,b]$.

Claim each $p \in \mathbb{P}_{2n+1}$ can be written $p = L_n p + q_{n+1} q$ for some $q \in \mathbb{P}_n$

since $p - L_n p = 0$ at $\{x_j\}$. so q can have at most $(2n+1)-(n+1) = n$ zeros.

So $\int p(x)dx = \int (L_n p)\,dx + \int q_{n+1} q\,dx = \sum w_k p(x_k)$ since interpolatory. ($\int q_{n+1}q\,dx$ = 0 orthog.)

Here start:

So we need to construct, for each n, q_{n+1} orthog to all $\mathbb{P}_n$. This is possible:

Lemma 9.15 $\exists$ unique seq. (q_n) with $q_0 = 1$ and $q_n(x) = x^n + r_{n-1}(x)$ (x^n monomial, r_{n-1} in $\mathbb{P}_{n-1}$)

which are orthog. set $q_n \perp q_m$ $n \ne m$, and $\mathrm{Span}\{q_0, \ldots q_n\} = \mathbb{P}_n$.

pf. construct by Gram-Schmidt.

$q_0 = 1$ $\quad q_1 = x - \frac{\int x q_0}{\int q_0^2} = x$, $\quad q_2 = x^2 - \frac{\int x^2 q_0}{\int q_0^2} - \frac{\int x^2 q_1}{\int q_1^2}$ ... etc ...

They are 'Legendre polynomials'.

We also need all zeros of q_n to be in $[a,b]$:

Lemma 9.16 q_n has n simple zeros in $[a,b]$.

$\int q_n = 0$ by $q_n \perp q_0$, $n>0$, so q_n has ≥ 1 zeros in $[a,b]$, call them $x_1 \cdots x_m$.

Suppose $m<n$, $r_m(x) := \prod_{j=1}^{m}(x-x_j) \in \mathbb{P}_{n-1}$ so is $\perp q_n$.

but $\int r_m q_n \neq 0$ since $r_m q_n$ has fixed sign & is $\not\equiv 0$.

$\Rightarrow$ contradiction. $\Rightarrow m=n$.

Thm 9.17: for each $n=0,1,\dots$ $\exists$ unique Gauss. quad form order n, with ~~zeros~~ nodes given by zeros of $(n+1)^{th}$ orthog. poly. q_{n+1}.

- Claim:
  $2n+1$ is highest poss. degree any quad. can achieve. Why? $p = \prod_{j=0}^{n}(x-x_j)^2 \in \mathbb{P}_{2n+2}$ vanishes at x_j, $Q_n(p)=0$ but $Q(p)>0$.

PS, This all generalizes to a weight function $(f,g) = \int_a^b \omega f g \, dx$, $\omega>0$ so poly's q_n ω-orthog, and $Q_n(f) = \int f(x)\omega(x)dx$. ... useful for f with singularities (move them onto ω).

~~[Gaussian Weights are positive. (Thm 9.18)~~

~~pf. $f_k(x) := \prod_{j\neq k}(x-x_j)^2$, eval. $f_k(x_j) = \begin{cases} 0 & j\neq k \\ [q'_{n+1}(x_k)]^2 & j=k \end{cases}$~~

~~so $\sum w_j f_k(x_j) = w_k[q'_{n+1}(x_k)]^2$~~

~~$= \int_a^b f_k(x)\,dx$ since $f_k \in \mathbb{P}_{2n}$ integr. exactly~~

~~>0 since $f\geq 0$. so $w_k \geq 0$.~~

Proves Gauss. quad. convergent.

~~Why care? means they cannot be wildly oscillating & large: recall 1 is integrated exactly so $\sum_{k=0}^{n} w_k = b-a$.~~ ~~→ lets see why.~~

This ~~§9.2 Convergence of Quadrature.~~ simpler ↓

Thm: Gaussian weights non-negative.

pf: (Stewart) $l_k(x_j) = \delta_{jk}$ so $l_k^2(x_j) = \delta_{jk}$ too

$0 < \int l_k^2(x)\,dx = \sum_{j=0}^{n} w_j l_k^2(x_j) = w_k$.

integrand >0. $l_k^2 \in \mathbb{P}_{2n}$

$\Rightarrow$ (Q_n) convergent.

~~How calc in practise? code gauss.m: eigvals of tridiag matrix~~

End of Lec 6.