Lec 1 M116 9/24/08 ①

Syllabus.

Topic Course is crash course in selected parts of Num. Anal + focusing on boundary integral eqns & global approximation; areas I love.

Mixture of analysis & coding, implementing stuff. Practical tools for your life as scientist.

Syllabus: HW: weekly, webpage

books.

computer — teach you how to code efficiently, debug as you go. : Jon Brown. Matlab, optional but rec.

~~topics~~. project. go further w/ something, or apply to problems, or new PDE.

grading: HW + final project. roughly equal. No exam.

Grad vs ugrad course: depth, more initiative from you, more flexibility from me. I (& Jon) need to hear what you're stuck on.

Num. Anal? LNT essay highlights. (handout.) { impact of numerical computation: algorithms key. NA disasters link. goal: understand math behind algorithms, code your own.

- PDEs in this course: 2 of the 3 big linear PDEs of math & physics.

1) Laplace eqn. $\Delta u = 0$, $u(\vec{x}) = u(x_1, x_2 \cdots x_n)$, $\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$

BVP: 

$\Delta u = 0$ in Ω

$u|_{\partial\Omega} = f$ given.

or in exterior region, $\mathbb{R}^n \setminus \bar{\Omega}$, with some decay at ∞, eg. $u(x) \to 0$ as $|x| \to \infty$ (in 3d)

applications: website resonhtml. pics. electrostatic potential. (f = applied voltages)

conformal mapping: u = harmonic function

2) Helmholtz eqn. $(\Delta + k^2)u = 0$ k = wavenumber. waves of constant freq.

short wavelength = large k.

comes from wave eqn: $\tilde{u}(\vec{x}, t)$: $\Delta \tilde{u} - \frac{1}{c^2}\tilde{u}_{tt} = 0$ c = wave speed.

assume const freq. $\tilde{u}(\vec{x}, t) = u(\vec{x}) e^{-i\omega t}$ (freq.) sub. into WE; $\tilde{u}_{tt} = (-i\omega)^2 \tilde{u}$

giving $(\Delta + \frac{\omega^2}{c^2}) u = 0$, $k = \frac{\omega}{c}$.

scattering $(\Delta + k^2) u = 0$ $\mathbb{R}^2 \setminus \bar{\Omega}$, $u|_{\partial\Omega} = f$ (given by incident field)

acoustics, EM (radar) : show pics. $\frac{\partial u}{\partial n} = 0$.

generalization to Maxwell eqns.

3) Laplace Eigenvalue prob: find nontrivial modes u_j & eigenfrequencies E_j

$$\begin{cases} (\Delta + E_j) u_j = 0 & \text{in } \Omega \\ u_j|_{\partial\Omega} = 0 \end{cases}$$

compact domain in $\mathbb{R}^n$.

apps: resonances of cavities, acoustic, EM, quantum

Missing: heat eqn, Stokes eqn. (fluids), Navier-Stokes (nonlin fluids).

Overall approach: push problem to the boundary $\partial\Omega$.

70 mins

to get to PDE will need { • lin. algebra (numer.) ... i) almost all num. PDE boils down to this. ii) essential background. • bit of rounding errors. }

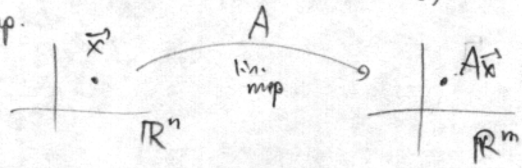
Numerical Lin. Alg

← Tref. & Bau book.

solving lin. sys. $A\vec{x} = \vec{b}$: stability & SVD.   stop me if stuck.

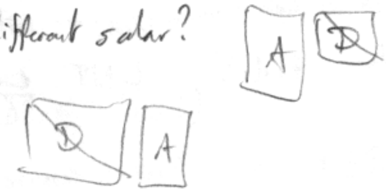
Lin Alg. recap.

$A \in \mathbb{C}^{m \times n}$ matrix



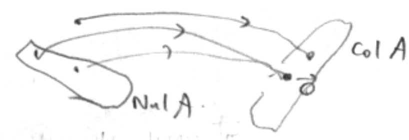
$A\vec{x}$ is lin. comb. cols $\vec{a}_j$ of A w/ coeffs $(x_1 \ldots x_n)$

how combine A w/ diag. matrix if want to mult. each {col. by different scalar? / row?}



Spaces: $\text{Col}\,A = \text{span}\{\vec{a}_j\} \subset \mathbb{R}^m$

$\text{Nul}\,A$ = all vectors which A kills $= \{\vec{x} : A\vec{x} = 0\} \subset \mathbb{R}^n$



$\text{rank}(A) = \dim \text{Col}\,A$ (= # of pivots) $\leq \min(n,m)$

Say A 'tall' $(m \geq n)$: what needs to hold s.t. A map is one-to-one?

each $\vec{y} = A\vec{x}$ must be unique lin. comb. of $\{\vec{a}_j\}$, $\Rightarrow$ $\{\vec{a}_j\}$ must be lin. indep. $\Rightarrow$ $\dim \text{Col}\,A = n$

also converses.

Thm: $(m \geq n)$ A full rank $\Leftrightarrow$ map 1-1. (soln. to $Ax=b$ unique if exists).

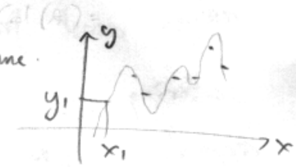
Square $(n=m)$: full rank $\Leftrightarrow$ A^{-1} exists st $AA^{-1} = A^{-1}A = I$.

then soln. $\vec{x} = A^{-1}\vec{b}$ is unique vec of coeffs. of expansion of $\vec{b}$ in basis of cols of A

Application: polynomial approximation: let $\{x_j\}_{j=1\ldots n}$ be distinct numbers. Claim: matrix A w/ elements $a_{ij} = x_i^{j-1}$ $i,j = 1 \ldots n$ is nonsingular.

How does A arise? Say have data $(x_j, y_j)_{j=1\ldots n}$ points in plane.

What is $n-1^{th}$ degree polynomial passing through data?



$p_{\vec{c}}(x) = c_0 + c_1 x + c_2 x^2 + \ldots c_{n-1} x^{n-1}$

Lin. eqns: $p(x_j) = y_j \quad \forall j = 1 \ldots n$

ie
$$\begin{cases} c_0 + c_1 x_1 + c \ldots c_{n-1} x_1^{n-1} = y_1 \\ c_0 + c_1 x_2 \ldots c_{n-1} x_2^{n-1} \quad\; y_2 \\ \vdots \\ c_0 + c_1 x_n \quad\; c_{n-1} x_n^{n-1} = y_n \end{cases}$$

ie $A\vec{c} = \vec{y}$

called Vandermonde matrix (square $n \times n$)

Suppose $\vec{c} \neq \vec{c}'$ were 2 such solutions.

Then $p_{\vec{c}}(x) - p_{\vec{c}'}(x)$ is nontrivial degree-$(n-1)$ poly, which must vanish at each x_j

ie have n distinct roots $\Rightarrow$ impossible, so $\vec{c}$ is unique $\Rightarrow$ A full rank.

70 mins.

Interlude : Matlab knows this stuff too:

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ rank(A) = 2. $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \\ 5 & 10 \end{bmatrix}$ rank(A) = 1. null(A) = $\begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix}$

Vandermonde $\left\{ \begin{array}{l} \texttt{x = -1:0.1:1;} \\ \texttt{A = vander(x);} \\ \texttt{A} \end{array} \right.$ ← check, hard to view #s

ways to show A : imagesc(A) colorbar.
spy(A).
plot(A) ← graphs each col. plot(x, A) uses correct x values

flip A in j axis : A = A(:, end:-1:1);

rank(A) = 21.
a = 30. -1:0.07:1
ok, try n = 40. : rank(A) = 36. ?why?
n = 100. 36 !

why? numerical rank ≠ theoretical rank

Start Lec. 2 :

Need more theory : orthogonality

A^* hermitian transpose. $(A^*)_{ij} = \overline{A_{ji}}$ ← c.c.

$\vec{x}$ col vec, $\vec{x}^*$ row vec.

inner prod. $\vec{x}^*\vec{y} = [\longrightarrow]\,[\downarrow] = \sum_{i=1}^{m} \bar{x}_i y_i$ $(AB)^* = B^*A^*$, $(A^{-1})^* = (A^*)^{-1}$ (prove it) (choose $B = A^{-1}$)

2-norm $\|x\|_2 = \sqrt{\vec{x}^*\vec{x}}$: norms have $\left\{ \begin{array}{l} \text{i) } \|\alpha x\| = |\alpha|\,\|x\| \\ \text{ii) } \|x\| = 0 \Rightarrow x = 0 \\ \text{iii) } \|x+y\| \le \|x\| + \|y\| \text{ . tri.} \end{array} \right.$ 2-norms also $|x^*y| \le \|x\|\|y\|$ Cauchy-Schwarz ineq.

↖ we'll usually drop the 2.

orthog. $\vec{x}^*\vec{y} = 0$.

Thm: vectors in an orthog. set are L.I. (prove it).

⇒ m orthog. vecs. in $\mathbb{C}^m$ form basis. if unit length, an o.n.b.

$\vec{q}_j$, $j = 1 \cdots m$ = o.n.b., stack in cols of Q, then $Q^{-1} = Q^*$ ie Q unitary (real-valued: orthogonal).

why? $(Q^*Q)_{ij} = i[\longrightarrow]\,\overset{j}{[\downarrow]} = q_i^* q_j = \delta_{ij}$

so $Q^*Q = I$

so $Q^*\vec{b}$ is coeffs of expansion of $\vec{b}$ in o.n.b. $\{q_i\}$ ⇝ no inverse req'd ⇒ nice.

$\|Qx\| = \sqrt{(Qx)^*(Qx)} = \sqrt{x^*\cancel{Q^*Q}x} = \|x\|$ so Q transformation preserves lengths. (if det Q = 1, Q real, it's rigid rotn.)

Matrices have 2-norms too! ⇝ guess meaning?

$\|A\|$ is smallest number C s.t. $\|Ax\| \le C\|x\|$ $\forall x \in \mathbb{C}^n$, ie max growth factor of a vector.

$\|A\| := \sup_{x \ne 0} \frac{\|Ax\|}{\|x\|}$ ← vec 2-norms., matrix norm induced by vec 2-norms.
$= \sup_{\|x\|=1} \|Ax\|$ ⇜ the longest a unit vector can become

What is 2-norm of diag matrix $\begin{pmatrix} a_{11} & & \\ & a_{22} & \\ & & \ddots \end{pmatrix}$? $\max |a_{jj}|$

WS if not time for other. — 'rank-1 matrix' $A = uv^*$ — 'outer-product' of 2 vectors (I)

i) why is $\mathrm{rank}(A) = 1$?

ii) compute 2-norm: $\|Ax\| = \|uv^*x\| = \underbrace{|v^*x|}_{\text{scalar}}\,\|u\| \le \|u\|\|v\|\|x\|$ C-S

so $\|A\| = \|u\|\|v\|$ — is equality? yes, $x = v$.

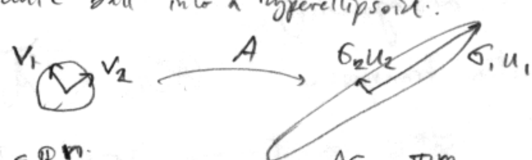
submultiplicative: $\|AB\| \le \|A\|\|B\|$ — pf. $\|A(Bx)\| \underset{\text{why?}}{\le} \|A\|\|Bx\| \underset{\text{why?}}{\le} \|A\|\|B\|\|x\|$

Thm 3.1 $\|QA\| = \|A\|$ (Q unitary) (unitary from left preserves matrix norm) — pf? $\|QAx\| \underset{\text{unit.}}{=} \|Ax\|$

true from right? $\|AQx\| \le \|A\|\|Qx\| = \|A\|\|x\|$ ✓

Sing. Val. Decomp. — as important as spectral decomp but few math. know it!

geom fact: every matrix $A \in \mathbb{C}^{m\times n}$ maps unit ball into a hyperellipsoid.



take $m \ge n$, full rank ($=n$)

left sing. vecs u_j are unit vecs along ellipsoid axes (are orthog.)

sing. vals $\sigma_1 \ge \sigma_2 \ge \cdots \ge \sigma_n > 0$ — lengths of principal semiaxes: $\sigma_1 = \|A\|_2$ is max stretch

right sing vecs v_j are preimages of $\sigma_j u_j$. (amazingly, also orthog!)

If $\mathrm{rank}(A) = r$, $\sigma_1, \ldots \sigma_r > 0$, while $\sigma_{r+1} = \cdots = \sigma_n = 0$

algebra: $Av_j = \sigma_j u_j$ $j = 1 \ldots n$.

$$A \underbrace{\begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}}_{V} = \underbrace{\begin{bmatrix} u_1 & \cdots & u_n \end{bmatrix}}_{\hat{U}} \underbrace{\begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_n \end{bmatrix}}_{\hat{\Sigma}}$$

V (square) nonsingular (since if not L.I. neither would $\sigma_j u_j$ be). $\hat{U}$ o.n. cols. $\hat{\Sigma}$ (square)

$\Rightarrow$ $\boxed{A} = \boxed{\hat{U}}\boxed{\hat{\Sigma}}\boxed{V^{-1}}$ — usual to complete $\hat{U} \to$ $m\times m$ U o.n.b. in which case $\boxed{A} = \boxed{U}\boxed{\Sigma}\boxed{V^{-1}}$

Defn. SVD: $\underset{m\times n}{A} = U\Sigma V^*$ — U unitary $m\times m$, Σ diag $m\times n$, V unitary $n\times n$, diag entries $\sigma_1 \ge \sigma_2 \cdots \ge \sigma_{\min(m,n)} \ge 0$

If can prove every A has SVD, will show: every matrix (even nonsymm, rect' ones) is rotation $\to$ stretching $\to$ rotation

ie every matrix is <u>diagonal</u> when expressed in correct basis for $\mathbb{R}^n$ & $\mathbb{R}^m$.

Cf. eigenvalue decomp. $A = VDV^{-1}$ which only for square, & regular (full set of evecs).

M116 Lec 2 - (2nd half). Tu ① 9/30/08

If A square & invertible, $A^{-1} = (U\Sigma V^*)^{-1} = V\Sigma^{-1}U^*$ ← form of SVD (if reverse cols of V, U, diag)

so A^{-1} has same SVD as A except w/ $U \leftrightarrow V$, $\sigma_j \leftrightarrow \sigma_j^{-1}$ diag entries σ_j^{-1}

Worksheet → needs $\sigma_1 = \|A\|_2$, $\sigma_m^{-1} = \|A^{-1}\|_2$

break

OH Fri 2-3 ?, Jon's

X-hr: v. important & useful: Taylor series convergence & plotting; learning Matlab skills relevant for class. esp. even if seasoned, incl. from another language. 3pm Wed. 201

HWs via LaTeX?

Proof of Existence of SVD: skip (grads read).

define $\sigma_1 = \|A\|_2$ $\partial B(0,1)$ cpt so $\sup_{\|x\|=1} \|Ax\|$ achieved somewhere, call it v_1, $\|v_1\|=1$

& $Av_1 = \sigma_1 u_1$ defines u_1, $\|u_1\|=1$

extend v_1 to o.n.b. for $\mathbb{C}^n$: $\{v_j\}$ $\xrightarrow{\text{stack in cols}}$ V_1 matrix

u_1 " $\mathbb{C}^m$: $\{u_j\}$ U_1

calc $U_1^* A V_1 =: S = \left[\begin{array}{c|c} \sigma_1 & w^* \\ \hline 0 & B \end{array}\right]$ where w is some vec $\in \mathbb{C}^{n-1}$

since $Av_1 \perp u_2, u_3, \ldots$

bound $\|S\|$ by $\left\| \left[\begin{array}{c|c} \sigma_1 & w^* \\ \hline 0 & B \end{array}\right] \begin{bmatrix} \sigma_1 \\ w \end{bmatrix} \right\| = \left\| \begin{bmatrix} \sigma_1^2 + \|w\|^2 \\ Bw \end{bmatrix} \right\| = \sqrt{(\sigma_1^2 + \|w\|^2)^2 + \|Bw\|^2} \geq \sigma_1^2 + \|w\|^2$

$= \sqrt{\sigma_1^2 + \|w\|^2} \left\| \begin{bmatrix} \sigma_1 \\ w \end{bmatrix} \right\|$ so $\|S\| \geq \sqrt{\sigma_1^2 + \|w\|^2}$

but since U, V unitary, $\|S\| = \|A\| = \sigma_1$ by Thm. 3.1 → so $\|w\| = 0$, $w = \vec{0}$, nice

Induction: $n=1$ or $m=1$ A has SVD trivially.

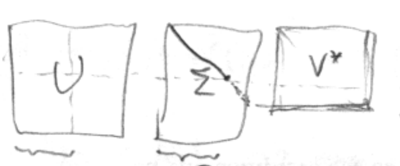
Now prove if B has SVD then A has one: $A = U_1 S V_1^* = \underbrace{U_1 \begin{bmatrix} 1 & \\ & U_2 \end{bmatrix}}_{\text{unit.}} \begin{bmatrix} \sigma_1 & \\ & \Sigma_2 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & \\ & V_2^* \end{bmatrix} V_1^*}_{\text{unit.}}$

B: $(m-1)\times(n-1)$, $U_2\Sigma_2V_2^*$; A: $(m\times n)$

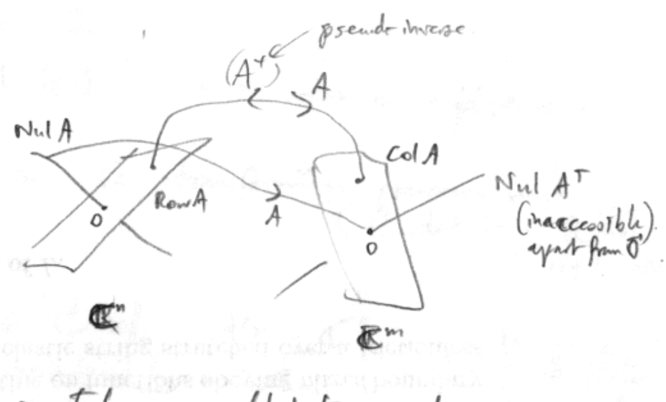
is SVD for A. QED.

Anatomy of SVD:

$r = \operatorname{rank} A = \#\{j : \sigma_j > 0\}$ since U, V full rank.



4 fundamental spaces.



square: $|\det A| = \prod_{j=1}^m \sigma_j$ (prove it)

numerical rank $r_\varepsilon = \#\{j : \sigma_j > \varepsilon\}$ where ε is tolerance related to rounding errors in CPU.

generally $\varepsilon = \sigma_1 \cdot$ (relative round err)