MATH 116 WORKSHEET: Singular values vs Eigenvalues

9/18/08
Barnett.

Consider the $m \times m$ matrix

$$A = \begin{bmatrix} 1 & 2 & & & \\ & 1 & 2 & & \\ & & 1 & 2 & \\ & & & \ddots & \ddots \\ & & & & \end{bmatrix}$$

with zeros everywhere apart from the diagonal & 1st super-diagonal.

Compute by hand:

a) eigenvalues of A

b) $\det A$

c) rank A

d) A^{-1}

e) Find a nontrivial upper bound on σ_m (the smallest singular value). You may use Matlab to evaluate the sing. vals. for eg $m = 10, 20$, etc, to get a hint. But you should prove your bound. [Hint: use A^{-1}]

This shows how different singular values & eigenvalues are for non-symmetric matrices!
(Trefethen, Num. Lin. Alg., EX 9.2)

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SOLUTIONS

Consider the $m \times m$ matrix

$$A = \begin{bmatrix} 1 & 2 & & & \\ & 1 & 2 & & \\ & & 1 & 2 & \\ & & & \ddots & \ddots \\ & & & & \end{bmatrix}$$

with zeros everywhere apart from the diagonal & 1st super-diagonal.

Compute by hand:

a) eigenvalues of A

A upper-triangular ⇒ diag elements are eig vals ⇒ $\lambda = 1$ m-fold degenerate.

b) det A

$\det A = \prod_{i=1}^{m} \lambda_i = 1$

c) rank A

full rank $r = m$ since $\det A \neq 0$.

d) A^{-1}

Take eg. $m=3$

linear system: $\overset{A}{\begin{bmatrix} 1 & 2 & \\ & 1 & 2 \\ & & 1 \end{bmatrix}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$

solve by back-substitution

$x_3 = y_3$ ③

$x_2 + 2x_3 = y_2$ ie $x_2 = y_2 - 2y_3$ ②

$x_1 + 2x_2 = y_1$ ie $x_1 = y_1 - 2x_2 = y_1 - 2y_2 + 4y_3$ ①

Eq's ①, ②, ③ give rows of A^{-1} so

$$A^{-1} = \begin{bmatrix} 1 & -2 & 2^2 \\ & 1 & -2 \\ & & 1 \end{bmatrix}$$

In general you continue to $\begin{bmatrix} 1 & & & \cdots \\ & 1 & & (-2)^2 \\ & & 1 & -2 \\ & & & 1 \end{bmatrix}$ top right element is $\underline{(-2)^{m-1}}$

e) Find a nontrivial upper bound on σ_m (the smallest singular value).

You may use Matlab to evaluate the sing. vals. for eg $m = 10, 20$, etc, to get a hint. But you should prove your bound. [Hint: use A^{-1}]

First use $\|A^{-1}\| = \frac{1}{\sigma_m}$ ← smallest sing. val. of A.

Why? $A = U\Sigma V^*$ so $A^{-1} = V\Sigma^{-1}U^*$ is a (permuted) SVD of A^{-1}. So largest sing. val. of A^{-1} is σ_m^{-1}. But this is also $\|A^{-1}\|$

Pick vector $\vec{x} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$ then $\|A^{-1}\vec{x}\| = \left\| \begin{bmatrix} (-2)^{m-1} \\ (-2)^{m-2} \\ \vdots \\ -2 \\ 1 \end{bmatrix} \right\| \geq 2^{m-1}\|x\|$

So, $\|A^{-1}\| \geq 2^{m-1}$ so $\sigma_m \leq \frac{1}{2^{m-1}}$ dies exponentially with matrix size!

This shows how different singular values & eigenvalues are for non-symmetric matrices!

(Trefethen, Num. Lin. Alg., EX 9.2) actual SVD on Matlab shows $\sigma_m \sim$