MATH 116 WORKSHEET : Lagrange interpolation, linear case

10/7/08
Barnett

Recall $\quad \ell_k(x) = \prod_{\substack{j=0 \\ j\neq k}}^{n} \frac{x-x_j}{x_k-x_j}$ $\qquad$ basis funcs $k=0,\dots,n$

$L_n f := \sum_{j=0}^{n} f(x_j)\, \ell_j(x)$ $\qquad$ approximating poly.

$f \in C^{n+1}[a,b]$

$f(x) - L_n f(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{j=0}^{n} (x-x_j)$ $\qquad$ error func, for some $\xi\in[a,b]$, for each $x\in[a,b]$

For $x_0 = a$, $x_1 = b$, $n=1$ linear case...

a) write out & interpret graphically
$\ell_0(x) =$
$\ell_1(x) =$
$(L_1 f)(x) =$

b) write out (error func.) $f(x) - L_1 f(x) = \dots$

c) give an upper bound on $\|f - L_1 f\|_\infty$ in terms of $\|f''\|_\infty$ and $h := b-a$:

L^∞-norm on $[a,b]$, ie $\sup_{x\in[a,b]} |\cdot|$

d) Now for general $n \geq 1$, & $x_j \in [a,b]$, $j=0,\dots,n$ give a similar such bound:

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~ SOLUTIONS ~

Recall

$$l_k(x) = \prod_{\substack{j=0 \\ j\neq k}}^{n} \frac{x-x_j}{x_k-x_j} \qquad \text{basis funcs } k=0,\cdots,n$$

$$L_n f := \sum_{j=0}^{n} f(x_j)\, l_j(x) \qquad \text{approximating poly.}$$

$f \in C^{n+1}[a,b]$

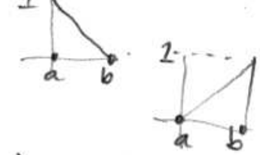
$$f(x) - L_n f(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{j=0}^{n} (x-x_j) \qquad \text{error func, for some } \xi\in[a,b], \text{ for each } x\in[a,b]$$

For $x_0 = a$, $x_1 = b$, $n=1$ linear case...

a) write out & interpret graphically

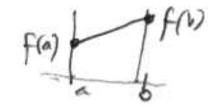
$$l_0(x) = \frac{x-b}{a-b}$$

$$l_1(x) = \frac{x-a}{b-a}$$



$$(L_1 f)(x) = \frac{1}{b-a}\big(f(a)(b-x) + f(b)(x-a) \big)$$

$=$



linear fit.

b) write out $f(x) - L_1 f(x)$ (error func.)

$$f(x) - L_1 f(x) = \frac{f''(\xi)}{2!}(x-a)(x-b) \quad \text{for some } \xi \in [a,b]$$

c) give an upper bound on $\|f - L_1 f\|_\infty$ in terms of $\|f''\|_\infty$ and $h := b-a$:

L^∞-norm on $[a,b]$, ie $\sup_{x\in[a,b]} |\cdot|$

$$|(x-a)(x-b)| \le \frac{h^2}{4} \quad \text{for all } x\in[a,b]$$

$$\text{so } \|f - L_1 f\|_\infty \le \frac{h^2}{8}\|f''\|_\infty$$

d) Now for general $n \ge 1$, & $x_j \in [a,b]$, $j=0,\cdots,n$ give a similar such bound:

A bound on monomial, $\left|\prod_{j=0}^{n}(x-x_j)\right| \le h^{n+1}$ $\forall x \in [a,b]$, since all we know is $x, x_0, \cdots x_n \in [a,b]$

$$\text{so } \|f - L_n f\|_\infty \le \frac{\|f^{(n+1)}\|_\infty}{(n+1)!} h^{n+1}$$

this is