MATH 116 WORKSHEET : simple Gaussian quadrature

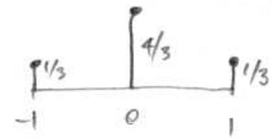
Barnett 10/9/08

Consider $[-1,1]$ integration. We'll fix $n=2$, ie 3 nodes.

Choosing nodes $x_0=-1$, $x_1=0$, $x_2=1$ we had Newton-Cotes quadrature

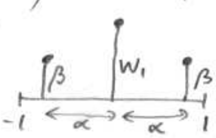
$$Q_2(f) = \sum_{k=0}^{3} w_k f(x_k) \quad \text{with} \quad w_0 = w_2 = \frac{1}{3}, \quad w_1 = \frac{4}{3}$$

This integrates degree-2 polys exactly



a) Explain why this happens to integrate x^m for m odd, exactly, too.

Now allow nodes to move inwards from ± 1, ie $x_0=-\alpha$, $x_1=0$, $x_2=\alpha$, and choose $w_0 = w_2 = \beta$



b) Use degree-0 exact integration to fix w_1 :

c) Write exactness conditions for degree-2 :

and for degree-4 :

d) Solve these for α, β

e) Up to what polynomial order is integrated exactly now?

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— SOLUTIONS —

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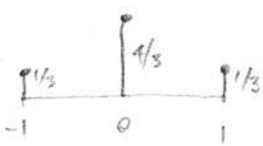
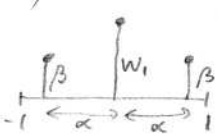


a) Explain why this happens to integrate x^m for m odd, exactly, too.

$$\int_{-1}^{1} x^m dx = 0 \text{ for } m \text{ odd}, \; \& \; \tfrac{1}{3}(-1)^m + \tfrac{4}{3}(0)^m + \tfrac{1}{3}1^m = 0 \Rightarrow \text{exact!}$$

Now allow nodes to move inwards from ± 1, ie $x_0=-\alpha$, $x_1=0$, $x_2=\alpha$

and choose $w_0 = w_2 = \beta$



b) Use degree-0 exact integration to fix w_1 : $\beta + w_1 + \beta = \int_{-1}^{1} dx = 2$.

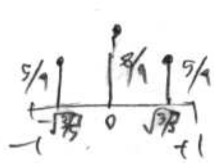
c) Write exactness conditions for degree-2 : $\beta\alpha^2 + \cancel{w_1(0)^2} + \beta\alpha^2 = \int_{-1}^{1} x^2 dx = 2/3$

and for degree-4 : $\beta\alpha^4 + 0 + \beta\alpha^4 = \int_{-1}^{1} x^4 dx = 2/5$

d) Solve these for α, β divide : $\alpha^2 = 3/5$, $\alpha = \sqrt{3/5}$

so $2\beta \cdot 3/5 = 2/3$, $\beta = 5/9$, $w_1 = 8/9$.

e) Up to what polynomial order is integrated exactly now?



$p \in \mathbb{P}_5$ is ~~exact~~ Note $5 = 2n+1$ $(n=2)$.

since you did 0, 2, 4, and all odd are exact.