

Math 116 Final Project

Yong Su

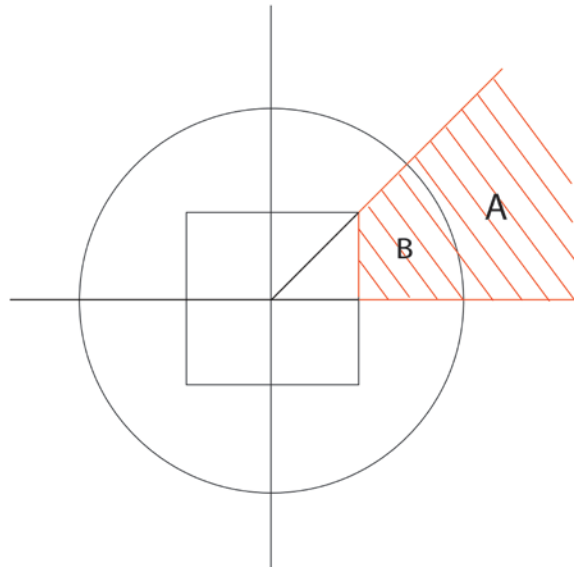
“Capacitance of a Unit Square”

The capacitance of a unit square Ω on the plane \mathbb{R}^2 is defined as $C = -\int_{\partial\Omega} \frac{\partial u}{\partial n} ds$, where u satisfies the following Laplace equation with Dirichlet boundary data:

- $\Delta u = 0$ in $\mathbb{R}^2 \setminus \partial\Omega$
- $u|_{\partial\Omega} = 0$
- $u(x) \sim \log|x| + O(1)$ as $|x| \rightarrow \infty$

For the purpose of this project, we only focus on how to get an accurate numerical solution of u and we will adopt Method of Particular Solutions (MPS)

If we naively use the same basis functions outside the square, we will get into trouble finding a convergence when we approach the corner of the square because the corner is singular (the normal on the corner is also ill-defined). Instead we place an artificial circle outside the square and separately solve the Laplace equation inside and outside the circle using MPS after which we can match up on the boundaries. Due to the geometric symmetry of the square and the circle, we only need to look at the red-shaded region illustrated below:



First, let us consider the solution in region A, which is outside the circle.

WLOG, write: $v(r, \theta) = u(r, \theta) - \log(r)$ (or simply, $v = u - \log r$)

Then v must satisfy the following Laplace equation:

- $\Delta v = 0$ in $\mathbb{R}^2 \setminus \partial\Omega$
- $v|_{\partial\Omega} = -\log r$
- $v \sim O(1)$ as $r \rightarrow +\infty$

Since v is harmonic at infinity, we use basic functions in the form $r^{-n} \cdot \sin(n\theta)$ or $r^{-n} \cdot \cos(n\theta)$ (Note we have used the fact that outside the disk, a Laurent expansion is complete.)

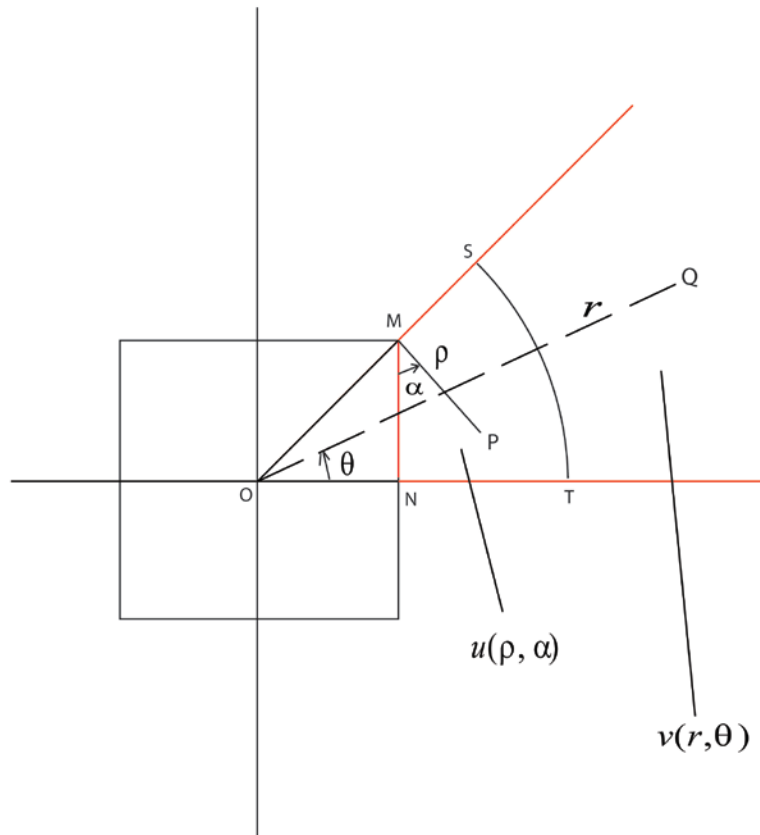
Also, in order to preserve the symmetry of the square and the artificial circle, we need these basis functions to appear symmetric even on the line $\theta = 0$ and $\theta = \pi/4$. Thus the only basis functions we can use are of the form $\xi_n(r, \theta) = r^{-4n} \cdot \cos(4n\theta)$, $n \in \mathbb{Z}^+$.

In other words, $v(r, \theta) = \sum_{i=0}^N c_i \cdot \xi_i(r, \theta)$, where $\{c_i\}$ are undetermined coefficients.

Note that here it is important that we start with the index $i = 0$ instead of with the index $i = 1$. This is because at exterior, the function v really is $O(1)$ plus a harmonic function that dies at infinity. As a result, a constant term should be added into the set of basis functions.

Now we consider the solution in the region B.

To simplify the expression of u , we switch the polar coordination system (centered at O) to be centered at M, the top-right corner of the square, which can be illustrated as follows:



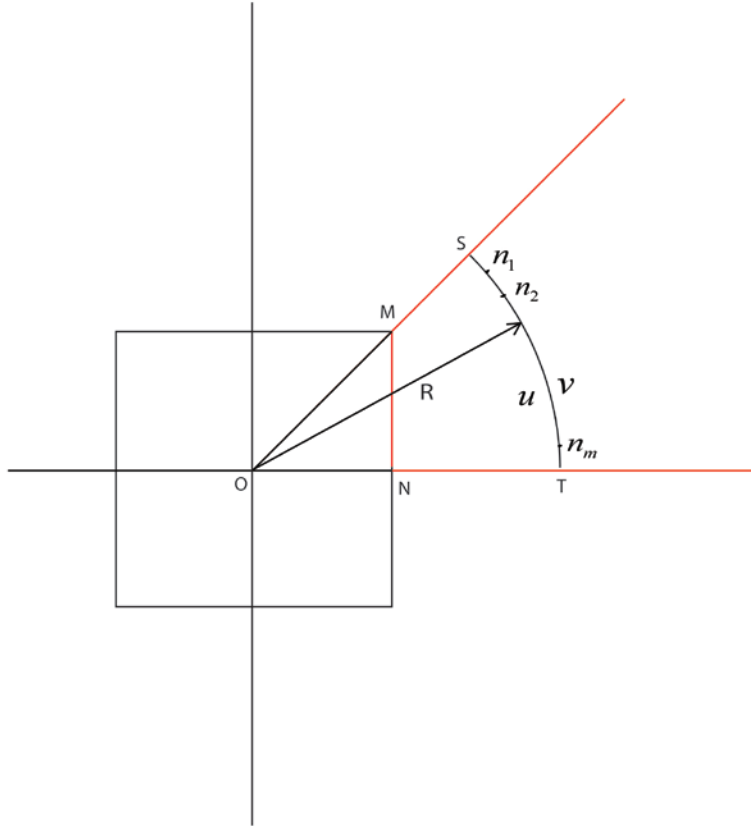
Since the solution u has to satisfy the boundary data on segment MN (i.e. $u(\rho, 0) = 0$) and it has to preserve symmetry according to the line $\alpha = \frac{3\pi}{4}$, we can only use basis functions of the form:

$$\varphi_n(\rho, \alpha) = \rho^{2(2n-1)/3} \cdot \sin(2(2n-1)\alpha / 3), \quad n \in \mathbb{Z}^+$$

Hence we can write $u(\rho, \alpha) = \sum_{i=1}^N a_i \cdot \varphi_i(\rho, \alpha)$, where $\{a_i\}$ are also undetermined coefficients.

To solve for coefficients $\{a_i\}$ and $\{c_i\}$, we need to match up all the boundary conditions. In fact, there are 3 boundary conditions we need to fit:

(1) If we place m nodes: n_1, n_2, \dots, n_m on the arc \widehat{ST} , u and v has to agree on these m nodes, with a jump $\log(R)$, where R denoted the radius of our imaginary circle. ($v = u - \log r$)



In other words, the following m linear equations must hold:

- $a_1\varphi_1(n_1) + a_2\varphi_2(n_1) + \dots + a_N\varphi_N(n_1) - c_1\xi_0(n_1) - c_2\xi_1(n_1) - \dots - c_N\xi_{N-1}(n_1) = \log(R)$
- $a_1\varphi_1(n_2) + a_2\varphi_2(n_2) + \dots + a_N\varphi_N(n_2) - c_1\xi_0(n_2) - c_2\xi_1(n_2) - \dots - c_N\xi_{N-1}(n_2) = \log(R)$
-
- $a_1\varphi_1(n_m) + a_2\varphi_2(n_m) + \dots + a_N\varphi_N(n_m) - c_1\xi_0(n_m) - c_2\xi_1(n_m) - \dots - c_N\xi_{N-1}(n_m) = \log(R)$

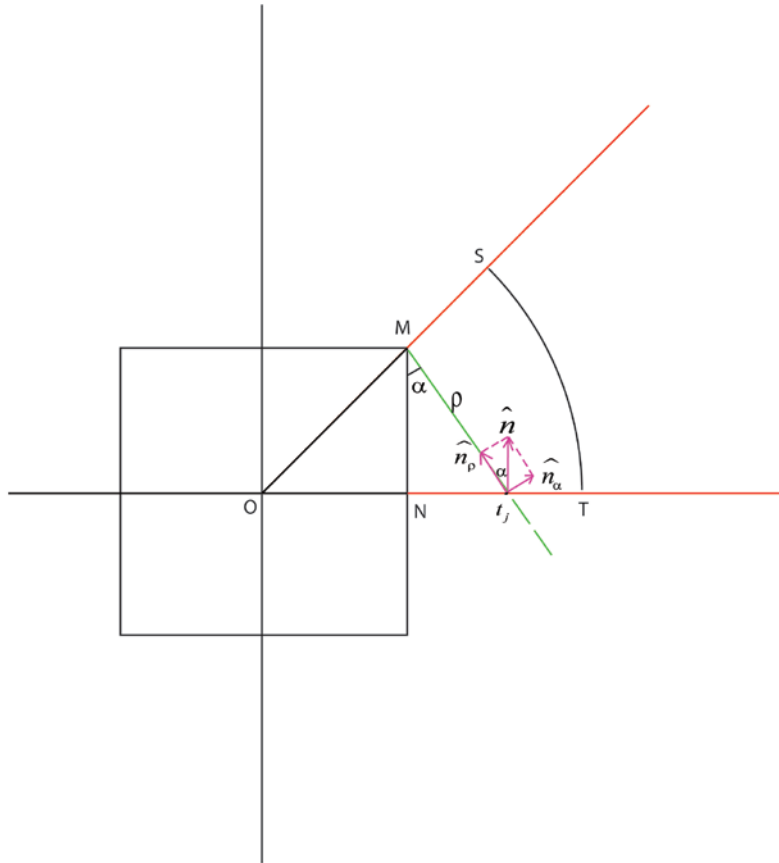
We have to be cautious that $\varphi_k(n_j)$ and $\xi_k(n_j)$ are actually using different coordination systems as we have discussed before.

For future convenience, we denote:

$$S_1 = \begin{pmatrix} \varphi_1(n_1) & \varphi_2(n_1) & \cdot & \cdot & \varphi_N(n_1) & -\xi_0(n_1) & -\xi_1(n_1) & \cdot & \cdot & -\xi_{N-1}(n_1) \\ \varphi_1(n_2) & \varphi_2(n_2) & \cdot & \cdot & \varphi_N(n_2) & -\xi_0(n_2) & -\xi_1(n_2) & \cdot & \cdot & -\xi_{N-1}(n_2) \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \varphi_1(n_m) & \varphi_2(n_m) & \cdot & \cdot & \varphi_N(n_m) & -\xi_0(n_m) & -\xi_1(n_m) & \cdot & \cdot & -\xi_{N-1}(n_m) \end{pmatrix}_{m \times 2N}$$

$b = (a_1, a_2, \dots, a_N, c_1, c_2, \dots, c_N)^T_{2N \times 1}$ is the coefficient matrix we want to solve.

(2) If we place m nodes t_1, t_2, \dots, t_m on the line segment NT, then the normal derivative of u on each of the node t_j should always equal zero because we want the basis functions



Notice that in polar coordination (ρ, α) , $\bar{\nabla} = \left(\frac{\partial}{\partial \rho}, \frac{1}{\rho} \cdot \frac{\partial}{\partial \alpha} \right)$

Hence, the normal derivative of a basis function φ_k on node t_j can be calculated as:

$$\frac{\partial}{\partial n} \varphi_k(\rho_j, \alpha_j) = -\cos \alpha \cdot \frac{\partial \varphi_k}{\partial \rho}(\rho_j, \alpha_j) + \sin \alpha \cdot \frac{1}{\rho} \frac{\partial \varphi_k}{\partial \alpha}(\rho_j, \alpha_j)$$

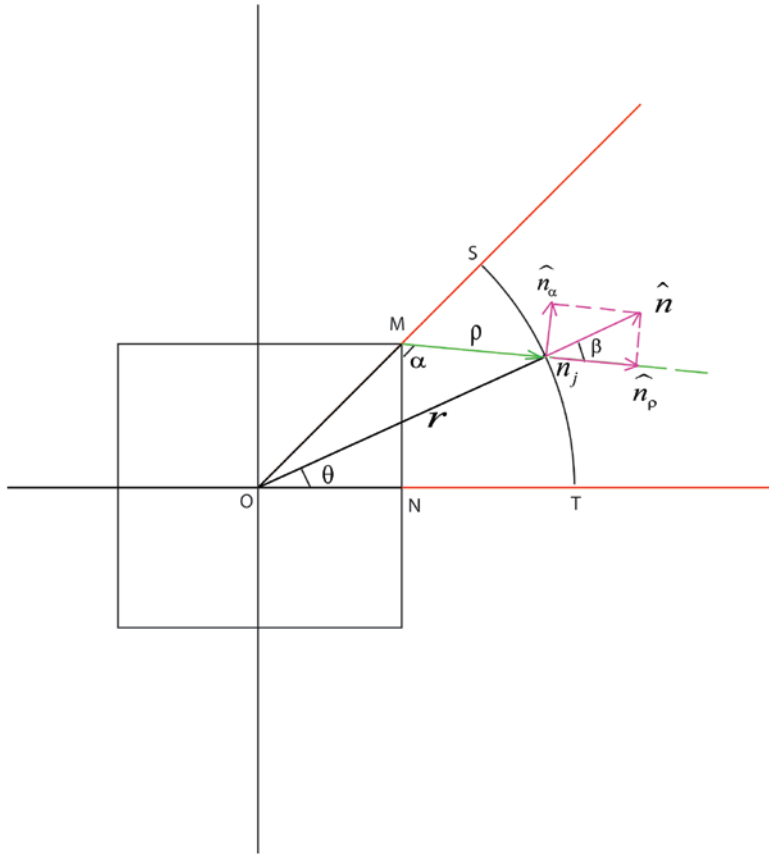
Therefore, we require the following linear equations to hold:

- $a_1 \varphi_{\hat{n}_1}(t_1) + a_2 \varphi_{\hat{n}_2}(t_1) + \dots + a_N \varphi_{\hat{n}_N}(t_1) = 0$
- $a_1 \varphi_{\hat{n}_1}(t_2) + a_2 \varphi_{\hat{n}_2}(t_2) + \dots + a_N \varphi_{\hat{n}_N}(t_2) = 0$
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- $a_1 \varphi_{\hat{n}_1}(t_m) + a_2 \varphi_{\hat{n}_2}(t_m) + \dots + a_N \varphi_{\hat{n}_N}(t_m) = 0$

Let us write:

$$S_2 = \begin{pmatrix} \varphi_{\hat{n}_1}(t_1) & \varphi_{\hat{n}_2}(t_1) & \dots & \varphi_{\hat{n}_N}(t_1) & 0 & 0 & \dots & 0 \\ \varphi_{\hat{n}_1}(t_2) & \varphi_{\hat{n}_2}(t_2) & \dots & \varphi_{\hat{n}_N}(t_2) & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \varphi_{\hat{n}_1}(t_m) & \varphi_{\hat{n}_2}(t_m) & \dots & \varphi_{\hat{n}_N}(t_m) & 0 & 0 & \dots & 0 \end{pmatrix}_{m \times 2N}$$

(3) Finally, the normal derivative of u and v must agree on the node n_j , $j = 1, 2, \dots, m$:



Same as what we have done in (2), the normal derivative of a basis function φ_k on node n_j can be calculated as:

$$\frac{\partial}{\partial n} \varphi_k(\rho_j, \alpha_j) = n_\rho \cdot \frac{\partial \varphi_k}{\partial \rho}(\rho_j, \alpha_j) + n_\alpha \cdot \frac{1}{\rho} \frac{\partial \varphi_k}{\partial \alpha}(\rho_j, \alpha_j)$$

Where $n_\rho = \cos \beta$, $n_\alpha = \sin \beta$ and $\beta = \pi/2 + \theta - \alpha$

On the other hand, since the unit normal derivative \hat{n} at node n_j agrees with the direction of \hat{r} at node n_j in the polar system (r, θ) , we have:

$$\frac{\partial}{\partial n} \xi_k(r_j, \theta_j) = 1 \cdot \frac{\partial \xi_k}{\partial r}(r_j, \theta_j) + 0 \cdot \frac{1}{r} \frac{\partial \xi_k}{\partial \theta}(r_j, \theta_j) = \frac{\partial \xi_k}{\partial r}(r_j, \theta_j)$$

Adding the jump log R into consideration (which yields a derivative of $1/R$), the following linear equations must hold:

- $a_1 \varphi_{\hat{n}_1}(n_1) + a_2 \varphi_{\hat{n}_2}(n_1) + \dots + a_N \varphi_{\hat{n}_N}(n_1) - c_1 \xi_{\hat{n}_0}(n_1) - c_2 \xi_{\hat{n}_1}(n_1) - \dots - c_N \xi_{\hat{n}_{(N-1)}}(n_1) = 1/R$
- $a_1 \varphi_{\hat{n}_1}(n_2) + a_2 \varphi_{\hat{n}_2}(n_2) + \dots + a_N \varphi_{\hat{n}_N}(n_2) - c_1 \xi_{\hat{n}_0}(n_2) - c_2 \xi_{\hat{n}_1}(n_2) - \dots - c_N \xi_{\hat{n}_{(N-1)}}(n_2) = 1/R$
-
- $a_1 \varphi_{\hat{n}_1}(n_m) + a_2 \varphi_{\hat{n}_2}(n_m) + \dots + a_N \varphi_{\hat{n}_N}(n_m) - c_1 \xi_{\hat{n}_0}(n_m) - c_2 \xi_{\hat{n}_1}(n_m) - \dots - c_N \xi_{\hat{n}_{(N-1)}}(n_m) = 1/R$

Again, write

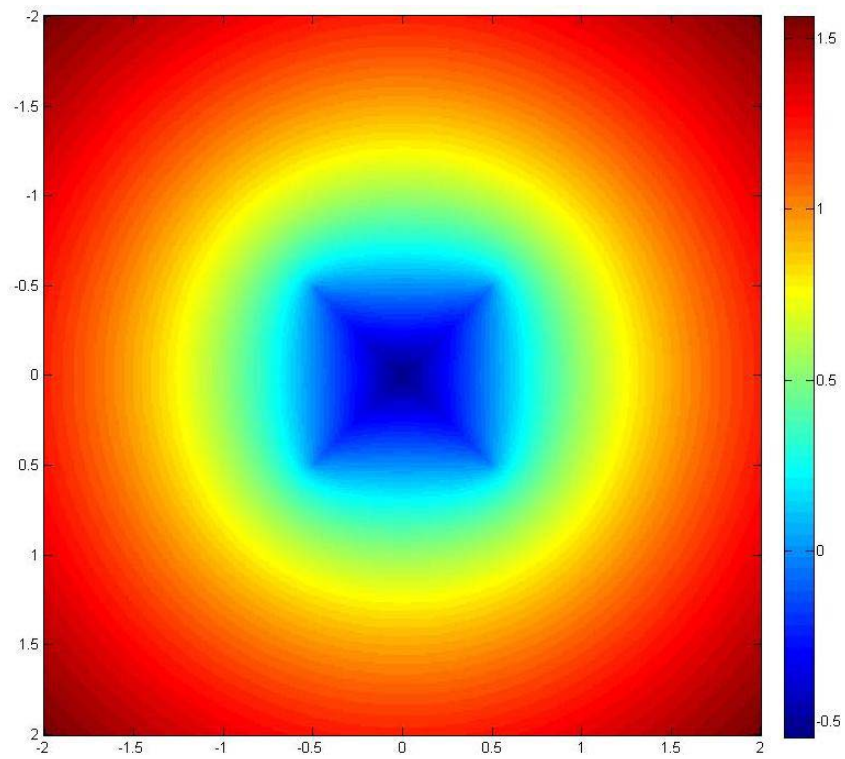
$$S_3 = \begin{pmatrix} \varphi_{\hat{n}_1}(n_1) & \varphi_{\hat{n}_2}(n_1) & \dots & \varphi_{\hat{n}_N}(n_1) & -\xi_{\hat{n}_0}(n_1) & -\xi_{\hat{n}_1}(n_1) & \dots & -\xi_{\hat{n}_{(N-1)}}(n_1) \\ \varphi_{\hat{n}_1}(n_2) & \varphi_{\hat{n}_2}(n_2) & \dots & \varphi_{\hat{n}_N}(n_2) & -\xi_{\hat{n}_0}(n_2) & -\xi_{\hat{n}_1}(n_2) & \dots & -\xi_{\hat{n}_{(N-1)}}(n_2) \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \varphi_{\hat{n}_1}(n_m) & \varphi_{\hat{n}_2}(n_m) & \dots & \varphi_{\hat{n}_N}(n_m) & -\xi_{\hat{n}_0}(n_m) & -\xi_{\hat{n}_1}(n_m) & \dots & -\xi_{\hat{n}_{(N-1)}}(n_m) \end{pmatrix}_{m \times 2N}$$

Now we have listed all the linear conditions that have to be imposed on u and v , it's time to solve for the whole linear system:

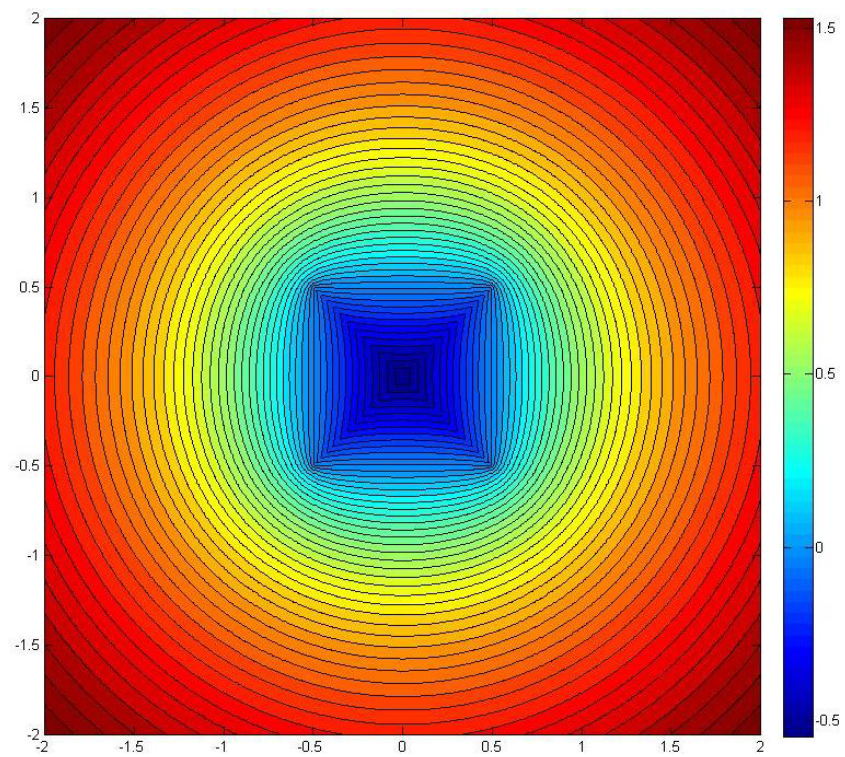
$$\text{Let } A = \begin{pmatrix} S_1 \\ S_2 \\ S_3 \end{pmatrix}_{3m \times 2N} \quad \text{and } c = \left(\log R, \log R, \dots, \log R, 0, 0, \dots, 0, \frac{1}{R}, \frac{1}{R}, \dots, \frac{1}{R} \right)^T_{3m \times 1}$$

Thus the linear system is just $Ab = c$ and we can solve for the coefficient vector b .

If we let $N=20$ and $m=3N$, here is the final solution of u :

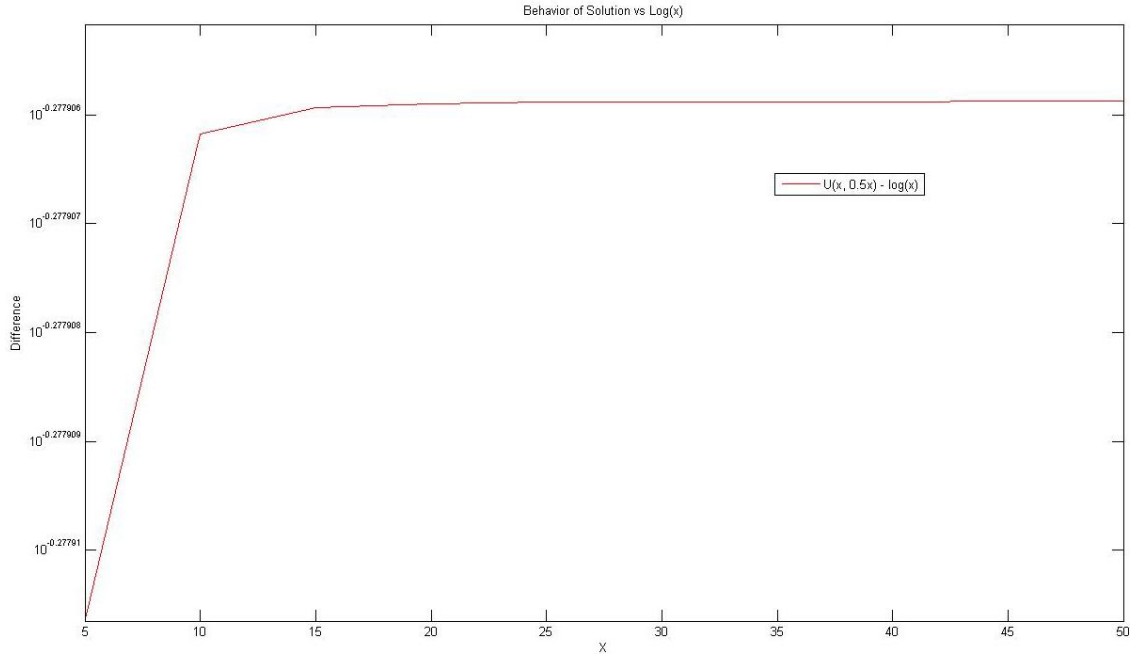


We can also make the contour plot for u :



These graphs clearly indicate that the boundary data is well satisfied.

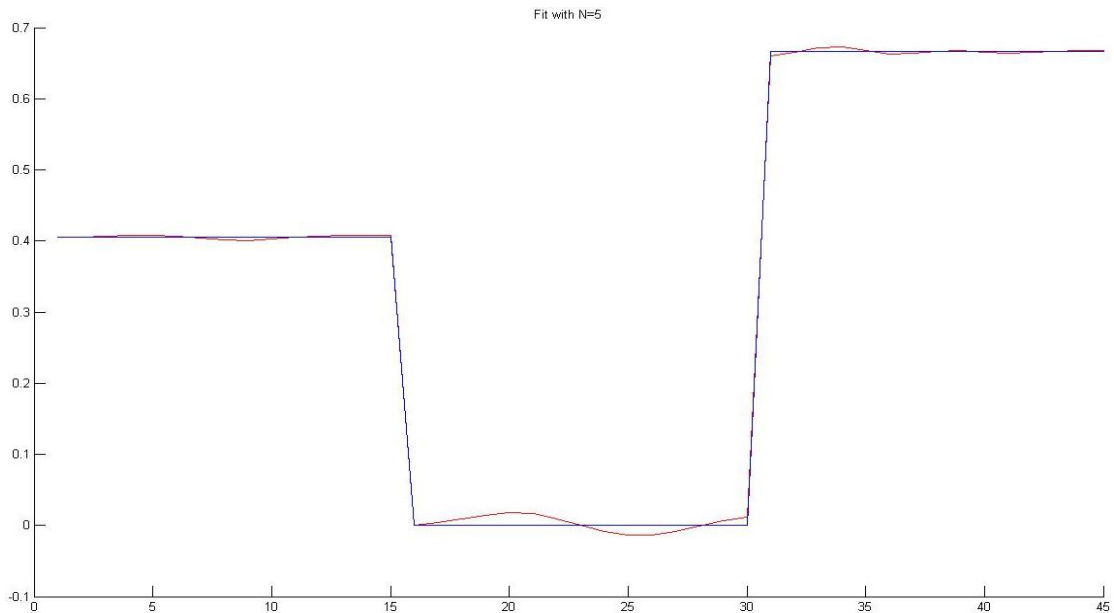
To verify that $u \sim O(1) + \log(r)$, WLOG we look at the values of $u(x, 0.5x) - \log x$ when x grows large (comparing to the size of our unit square):



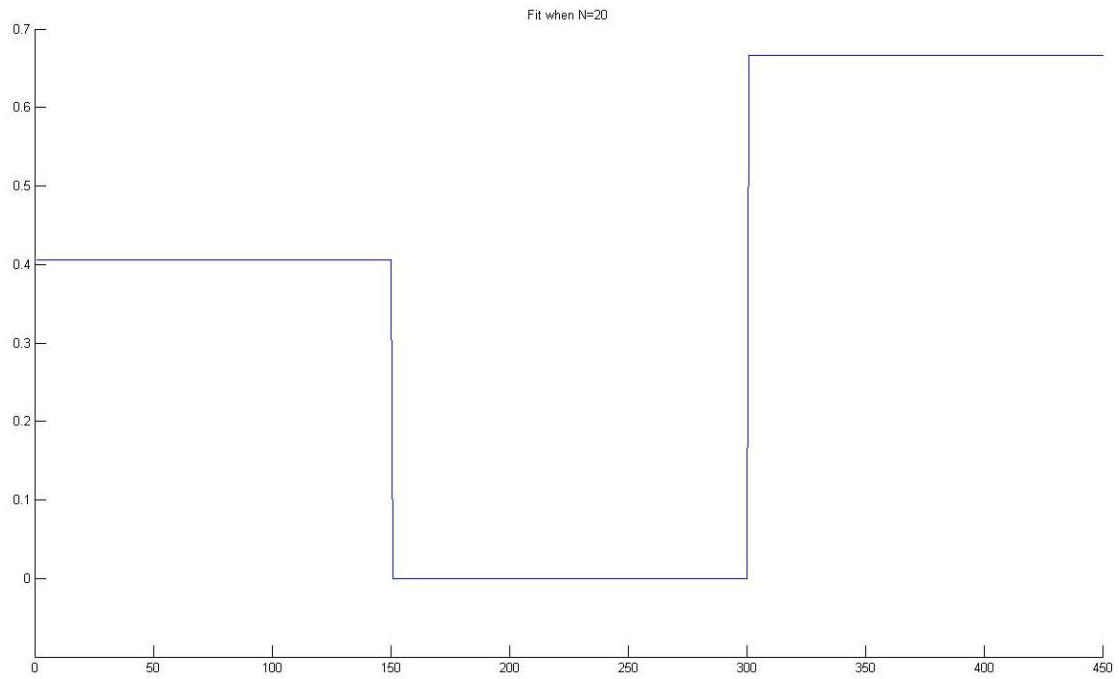
We see that the convergence is really fast.

Finally, another way to check the convergence of our solution is to plot $A \times \hat{b}$ vs c . Again, WLOG we examine the value of $u(1, 0.5)$ which is inside the circle.

When $N=5$, the fit appears like:



When $N=20$, the fit looks almost perfect:



In fact, if we plot the norm of $A \times \hat{b} - c$, we will get an exponential convergence:

